CHAPTER TWO
STRAIN

Learning objectives

1. Understand the concept of strain.
2. Understand the use of approximate deformed shapes for calculating strains from displacements.

How much should the drive belts (Figure 2.1*a*) stretch when installed? How much should the nuts in the turnbuckles (Figure 2.1*b*) be tightened when wires are attached to a traffic gate? Intuitively, the belts and the wires must stretch to produce the required tension. As we see in this chapter *strain* is a measure of the intensity of deformation used in the design against deformation failures.

(a)

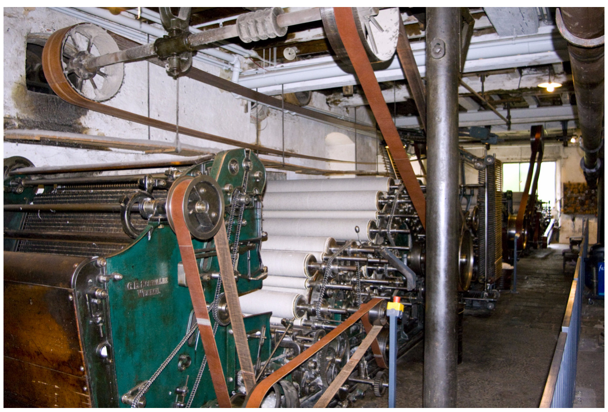


Figure 2.1 (a) Belt Drives (Courtesy Sozi). (b) Turnbuckles.

A change in shape can be described by the displacements of points on the structure. The relationship of strain to displacement depicted in Figure 2.2 is thus a problem in geometry—or, since displacements involve motion, a problem in kinematics. This relationship shown in Figure 2.2 is a link in the logical chain by which we shall relate displacements to external forces as discussed in Section 3.2. The primary tool for relating displacements and strains is drawing the body's approximate deformed shape. This is analogous to drawing a free-body diagram to obtain forces.

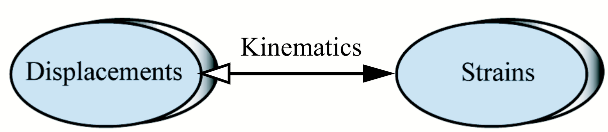


Figure 2.2 Strains and displacements.

2.1 DISPLACEMENT AND DEFORMATION

Motion of due to applied forces is of two types. (i) In rigid-body motion, the body as a whole moves without changing shape. (ii) In motion due to *deformation*, the body shape change. But, how do we decide if a moving body is undergoing deformation?

In rigid body, by definition, the distance between any two points does not change. In translation, for example, any two points on a rigid body will trace parallel trajectories. If the distance between the trajectories of two points changes, then the body is deforming. In addition to translation, a body can also rotate. On rigid bodies all lines rotate by equal amounts. If the angle between two lines on the body changes, then the body is deforming.

Whether it is the distance between two points or the angle between two lines that is changing, deformation is described in terms of the relative movements of points on the body. **Displacement** is the absolute movement of a point with respect to a fixed reference frame. **Deformation** is the relative movement with respect to another point on the same body. Several examples and problems in this chapter will emphasize the distinction between deformation and displacement.

2.2 LAGRANGIAN AND EULERIAN STRAIN

A handbook cost $L_0 = \$100$ a year ago. Today it costs $L_f = \$125$. What is the percentage change in the price of the handbook? Either of the two answers is correct. (i) The book costs 25% *more* than what it cost a year ago. (ii) The book cost 20% *less* a year ago than what it costs today. The first answer treats the original value as a reference: change $= [(L_f - L_0)/L_0] \times 100$. The second answer uses the final value as the reference: change $= [(L_0 - L_f)/L_f] \times 100$. The two arguments emphasize the necessity to specify the reference value from which change is calculated.

In the contexts of deformation and strain, this leads to the following definition: **Lagrangian strain** is computed by using the original undeformed geometry as a reference. **Eulerian strain** is computed using the final deformed geometry as a reference. The Lagrangian description is usually used in solid mechanics. The Eulerian description is usually used in fluid mechanics. When a material undergoes very large deformations, such as in soft rubber or projectile penetration of metals, then either description may be used, depending on the need of the analysis. We will use Lagrangian strain in this book, except in a few "stretch yourself" problems.

2.3 AVERAGE STRAIN

In Section 2.1 we saw that to differentiate the motion of a point due to translation from deformation, we need to measure changes in length. To differentiate the motion of a point due to rotation from deformation, we need to measure changes in angle. In this section we discuss *normal* strain and *shear* strain, which are measures of changes in length and angle, respectively.

2.3.1 Normal Strain

Figure 2.3 shows a line on the surface of a balloon that grows from its original length L_0 to its final length L_f as the balloon expands. The change in length $L_f - L_0$ represents the deformation of the line. *Average* normal strain is the intensity of deformation defined as a ratio of deformation to original length.

$$\boxed{\varepsilon_{av} = \frac{L_f - L_0}{L_0}} \tag{2.1}$$

where ε is the Greek symbol epsilon used to designate normal strain and the subscript av emphasizes that the normal strain is an average value. The following sign convention follows from Equation (2.1). Elongations ($L_f > L_0$) result in **positive normal strains**. Contractions ($L_f < L_0$) result in **negative normal strains**.

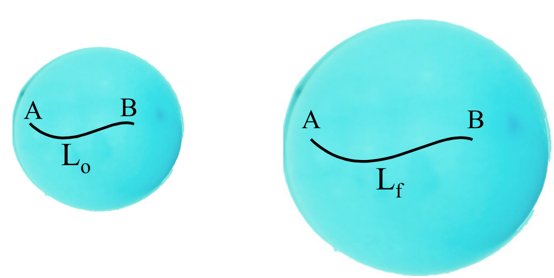
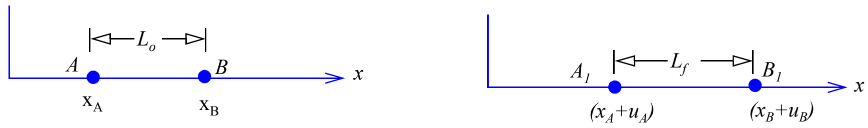


Figure 2.3 Normal strain and change in length.

An alternative form of Equation (2.1) is:

$$\boxed{\varepsilon_{av} = \frac{\delta}{L_0}} \tag{2.2}$$

where the Greek letter delta (δ) designates deformation of the line and is equal to $L_f - L_0$.



$$L_0 = x_B - x_A$$

$$L_f = (x_B + u_B) - (x_A + u_A) = L_o + (u_B - u_A)$$

Figure 2.4 Normal strain and displacement.

We now consider a special case in which the displacements are in the direction of a straight line. Consider two points A and B on a line in the x direction, as shown in Figure 2.4. Points A and B move to A_1 and B_1, respectively. The coordinates of the point change from x_A and x_B to $x_A + u_A$ and $x_B + u_B$, respectively. From Figure 2.4 we see that $L_0 = x_B - x_A$ and $L_f - L_0 = u_B - u_A$. From Equation (2.1) we obtain

$$\boxed{\varepsilon_{av} = \frac{u_B - u_A}{x_B - x_A}} \tag{2.3}$$

where u_A and u_B are the displacements of points A and B, respectively. Hence $u_B - u_A$ is the relative displacement, that is, it is the deformation of the line.

2.3.2 Shear Strain

Figure 2.5 shows an elastic band with a grid attached to two wooden bars using masking tape. The top wooden bar is slid to the right, causing the grid to deform. As can be seen, the angle between lines ABC changes. The measure of this change of angle is defined by shear strain, usually designated by the Greek letter gamma (γ). The average Lagrangian shear strain is defined as the change of angle from a right angle:

$$\boxed{\gamma_{av} = \frac{\pi}{2} - \alpha} \tag{2.4}$$

where the Greek letter alpha (α) designates the final angle measured in radians (rad), and the Greek letter pi (π) equals 3.14159 rad. Decreases in angle ($\alpha < \pi/2$) result in **positive shear strains**. Increases in angle ($\alpha > \pi/2$) result in **negative shear strains**.

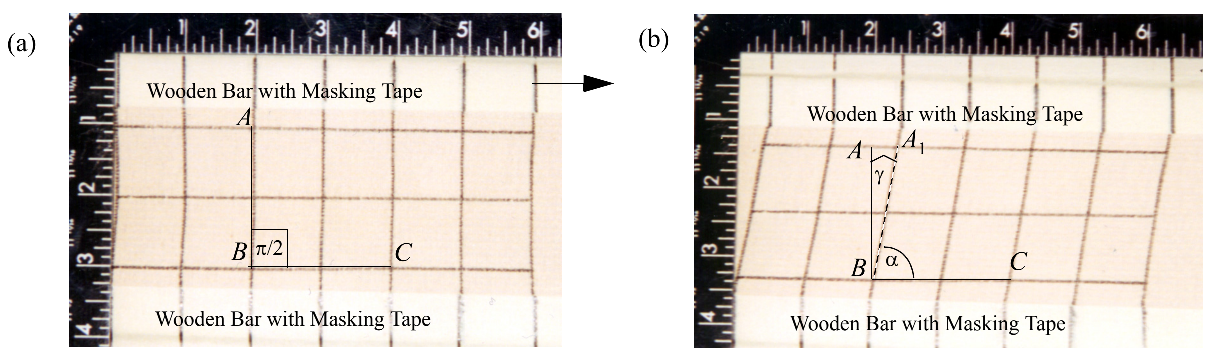


Figure 2.5 Shear strain and angle changes. (*a*) Undeformed grid. (*b*) Deformed grid.

2.3.3 Units of Average Strain

Equation (2.1) shows that normal strain is dimensionless, and hence should have no units. However, to differentiate average strain and strain at a point (discussed in Section 2.5), average normal strains are reported in units of length, such as in/in, cm/cm, or m/m. Radians are used in reporting average shear strains.

A percentage change is used for strains in reporting large deformations. Thus a normal strain of 0.5% is equal to a strain of 0.005. The Greek letter mu (μ) representing *micro* ($\mu = 10^{-6}$), is used in reporting small strains. Thus a strain of 1000 μ in/in is the same as a normal strain of 0.001 in/in.

EXAMPLE 2.1

The displacements in the x direction of the rigid plates in Figure 2.6 due to a set of axial forces were observed as given. Determine the axial strains in the rods in sections AB, BC, and CD.

$$u_A = -0.0100 \text{ in.} \qquad u_B = 0.0080 \text{ in.}$$
$$u_C = -0.0045 \text{ in.} \qquad u_D = 0.0075 \text{ in.}$$

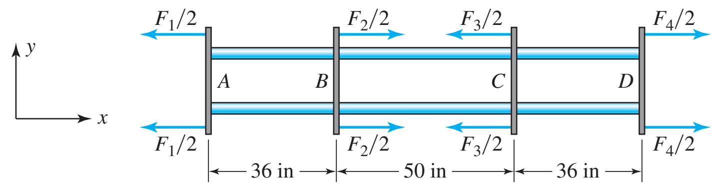


Figure 2.6 Axial displacements in Example 2.1.

PLAN

We first calculate the relative movement of rigid plates in each section. From this we can calculate the normal strains using Equation (2.3).

SOLUTION

The strains in each section can be found as shown in Equations (E1) through (E3).

$$\varepsilon_{AB} = \frac{u_B - u_A}{x_B - x_A} = \frac{0.018 \text{ in.}}{36 \text{ in.}} = 0.0005\frac{\text{in.}}{\text{in.}} \tag{E1}$$

ANS. $\varepsilon_{AB} = 500 \ \mu\text{in./ in.}$

$$\varepsilon_{BC} = \frac{u_C - u_B}{x_C - x_B} = \frac{-0.0125 \text{ in.}}{50 \text{ in.}} = -0.00025\frac{\text{in.}}{\text{in.}} \tag{E2}$$

ANS. $\varepsilon_{BC} = -250 \ \mu\text{in./ in.}$

$$\varepsilon_{CD} = \frac{u_D - u_C}{x_D - x_C} = \frac{0.012 \text{ in.}}{36 \text{ in.}} = 0.0003333\frac{\text{in.}}{\text{in.}} \tag{E3}$$

ANS. $\varepsilon_{CD} = 333.3 \ \mu\text{in./ in.}$

COMMENT

1. This example brings out the difference between the displacements, which were given, and the deformations, which we calculated before finding the strains.

EXAMPLE 2.2

A bar of hard rubber is attached to a rigid bar, which is moved to the right relative to fixed base A as shown in Figure 2.7. Determine the average shear strain at point A.

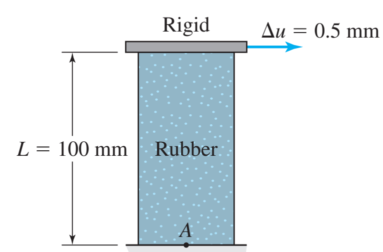


Figure 2.7 Geometry in Example 2.2.

PLAN

The rectangle will become a parallelogram as the rigid bar moves. We can draw an approximate deformed shape and calculate the change of angle to determine the shear strain.

SOLUTION

Point B moves to point B_1, as shown in Figure 2.8. The shear strain represented by the angle between BAB_1 is:

$$\gamma = \tan^{-1}\left(\frac{BB_1}{AB}\right) = \tan^{-1}\left(\frac{0.5 \text{ mm}}{100 \text{ mm}}\right) = 0.005 \text{ rad} \tag{E1}$$

ANS. $\gamma = 5000\mu\text{rad}$.

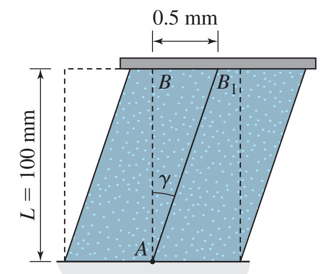


Figure 2.8 Exaggerated deformed shape.

COMMENTS

1. We assumed that line AB remained straight during the deformation in Figure 2.8. If this assumption were not valid, then the shear strain would vary in the vertical direction. To determine the varying shear strain, we would need additional information. Thus our assumption of line AB remaining straight is the simplest assumption that accounts for the given information.
2. The values of γ and tan γ are roughly the same when the argument of the tangent function is small. Thus for small shear strains the tangent function can be approximated by its argument.

EXAMPLE 2.3

A thin ruler, 12 in. long, is deformed into a circular arc with a radius of 30 in. that subtends an angle of 23° at the center. Determine the average normal strain in the ruler.

PLAN

The final length is the length of a circular arc and original length is given. The normal strain can be obtained using Equation (2.1).

SOLUTION

The original length $L_0 = 12$ in. The angle subtended by the circular arc shown in Figure 2.9 can be found in terms of radians:

$$\Delta\theta = \frac{(23^o)\pi}{180^o} = 0.4014 \text{ rads} \tag{E1}$$

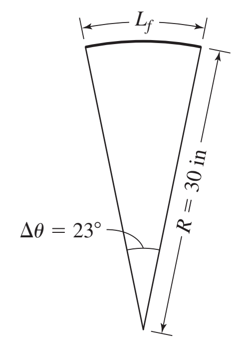


Figure 2.9 Deformed geometry in Example 2.3.

The length of the arc is:

$$L_f = R\Delta\theta = 12.04277 \text{ in.} \tag{E2}$$

and average normal strain is

$$\varepsilon_{av} = \frac{L_f - L_0}{L_0} = \frac{0.04277 \text{ in.}}{12 \text{ in.}} = 3.564(10^{-3})\frac{\text{in.}}{\text{in.}} \tag{E3}$$

ANS. $\varepsilon_{av} = 3564\ \mu\text{in./in.}$

COMMENTS

1. In Example 2.1 the normal strain was generated by the displacements in the axial direction. In this example the normal strain is being generated by bending.
2. In Chapter 6 on the symmetric bending of beams we shall consider a beam made up of lines that will bend like the ruler and calculate the normal strain due to bending as we calculated it in this example.

EXAMPLE 2.4

A belt and a pulley system in a VCR has the dimensions shown in Figure 2.10. To ensure adequate but not excessive tension in the belts, the average normal strain in the belt must be a minimum of 0.019 mm/mm and a maximum of 0.034 mm/mm. What should be the minimum and maximum undeformed lengths of the belt to the nearest millimeter?

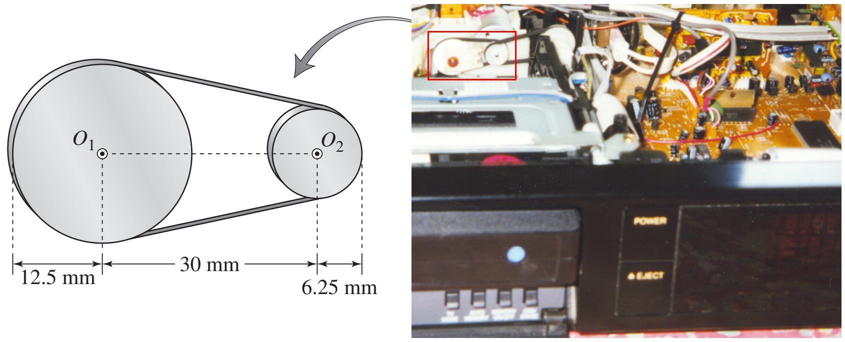


Figure 2.10 Belt and pulley in a VCR.

PLAN

The belt must be tangent at the point where it comes in contact with the pulley. The deformed length of the belt is the length of belt between the tangent points on the pulleys, plus the length of belt wrapped around the pulleys. Once we calculate the deformed length of the belt using geometry, we can find the original length using Equation (2.1) and the given limits on normal strain.

SOLUTION

We draw radial lines from the center to the tangent points A and B, as shown in Figure 2.11. The radial lines O_1A and O_2B must be perpendicular to the belt AB, hence both lines are parallel and at the same angle θ with the horizontal. We can draw a line parallel to AB through point O_2 to get line CO_2. Noting that CA is equal to O_2B, we can obtain CO_1 as the difference between the two radii.

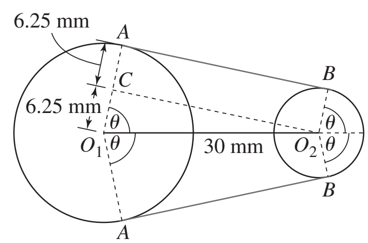


Figure 2.11 Analysis of geometry.

Triangle O_1CO_2 in Figure 2.11 is a right triangle, so we can find side CO_2 and the angle θ as:

$$AB = CO_2 = \sqrt{(30\text{ mm})^2 - (6.25\text{ mm})^2} = 29.342\text{ mm} \tag{E1}$$

$$\cos\theta = \frac{CO_1}{O_1O_2} = \frac{6.25\text{ mm}}{30\text{ mm}} \quad or \quad \theta = \cos^{-1}(0.2083) = 1.3609\text{ rad} \tag{E2}$$

The deformed length L_f of the belt is the sum of arcs AA and BB and twice the length AB:

$$AA = (12.5\text{ mm})(2\pi - 2\theta) = 44.517\text{ mm} \tag{E3}$$

$$BB = (6.25\text{ mm})(2\pi - 2\theta) = 22.258\text{ mm} \tag{E4}$$

$$L_f = 2(AB) + AA + BB = 125.46\text{ mm} \tag{E5}$$

We are given that $0.019 \le \varepsilon \le 0.034$. From Equation (2.1) we obtain the limits on the original length:

$$\varepsilon = \frac{L_f - L_0}{L_0} \le 0.034 \quad \text{or} \quad L_0 \ge \frac{125.46}{1 + 0.034}\text{ mm} \quad \text{or} \quad L_0 \ge 121.33\text{ mm} \tag{E6}$$

$$\varepsilon = \frac{L_f - L_0}{L_0} \ge 0.019 \quad or \quad L_0 \le \frac{125.46}{1 + 0.019}\text{ mm} \quad or \quad L_0 \le 123.1\text{ mm} \tag{E7}$$

To satisfy Equations (E6) and (E7) to the nearest millimeter, we obtain the following limits on the original length L_0:

ANS. $122\text{ mm} \le L_0 \le 123\text{ mm}$

COMMENTS

1. We rounded upward in Equation (E6) and downwards in Equation (E7) to ensure the inequalities.

2. Tolerances in dimensions must be specified for manufacturing. Here we have a tolerance range of 1 mm.
3. The difficulty in this example is in the analysis of the geometry rather than in the concept of strain. This again emphasizes that the analysis of deformation and strain is a problem in geometry. Drawing the approximate deformed shape is essential.

2.4 SMALL-STRAIN APPROXIMATION

In many engineering problems, a body undergoes only small deformations. A significant simplification can then be achieved by approximation of small strains, as demonstrated by the simple example shown in Figure 2.12. Due to a force acting on the bar, point P moves by an amount D at an angle θ to the direction of the bar. From the cosine rule in triangle APP_1, the length L_f can be found in terms of L_0, D, and θ:

$$L_f = \sqrt{L_0^2 + D^2 + 2L_0 D\cos\theta} = L_0\sqrt{1 + \left(\frac{D}{L_0}\right)^2 + 2\left(\frac{D}{L_0}\right)\cos\theta}$$

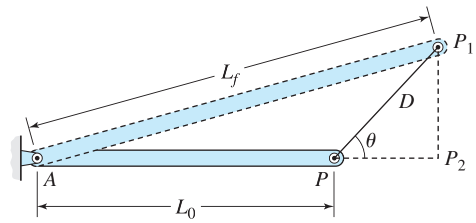


Figure 2.12 Small normal-strain calculations.

From Equation (2.1) we obtain the average normal strain in bar AP:

$$\varepsilon = \frac{L_f - L_0}{L_0} = \sqrt{1 + \left(\frac{D}{L_0}\right)^2 + 2\left(\frac{D}{L_0}\right)\cos\theta} - 1 \tag{2.5}$$

Equation (2.5) is valid regardless of the magnitude of the deformation D. Now suppose that D/L_0 is small. In such a case we can neglect the $(D/L_0)^2$ term and expand the radical by binomial[1] expansion:

$$\varepsilon \approx \left(1 + \frac{D}{L_0}\cos\theta + \dots + \dots\right) - 1$$

Neglecting the higher-order terms, we obtain an approximation for small strain in Equation (2.6).

$$\varepsilon_{\text{small}} = \frac{D\cos\theta}{L_0} \tag{2.6}$$

In Equation (2.6) the deformation D and strain are linearly related, whereas in Equation (2.5) deformation and strain are nonlinearly related. This implies that small-strain calculations require only a linear analysis, a significant simplification.

Equation (2.6) implies that in small-strain calculations only the component of deformation in the direction of the original line element is used. We will make significant use of this observation. Another way of looking at small-strain approximation is to say that the deformed length AP_1 is approximated by the length AP_2.

TABLE 2.1 **Small-strain approximation**

$\varepsilon_{\text{small}}$, [Equation (2.6)]	ε, [Equation (2.5)]	% Error, $\left(\frac{\varepsilon - \varepsilon_{\text{small}}}{\varepsilon}\right) \times 100$
1.000	1.23607	19.1
0.500	0.58114	14.0
0.100	0.10454	4.3
0.050	0.00512	2.32
0.010	0.01005	0.49
0.005	0.00501	0.25
0.001	0.00100	0.05

What is small strain? To answer this question we compare strains from Equation (2.6) to those from Equation (2.5). For different values of small strain and for $\theta = 45°$, the ratio of D/L is found from Equation (2.6), and the strain from Equation

[1]For small d, binomial expansion is $(1 + d)^{1/2} = 1 + d/2$ + terms of d^2 and higher order.

(2.5) is calculated as shown in Table 2.1. Equation (2.6) is an approximation of Equation (2.5), and the error in the approximation is shown in the third column of Table 2.1. It is seen from Table 2.1 that when the strain is less than 0.01, then the error is less than 1%, which is acceptable for most engineering analyses.

We conclude this section with summary of our observations.

1. Small-strain approximation may be used for strains less than 0.01.
2. Small-strain calculations result in linear deformation analysis.
3. Small normal strains are calculated by using the deformation component in the original direction of the line element, regardless of the orientation of the deformed line element.
4. In small shear strain (γ) calculations the following approximations may be used for the trigonometric functions: $\tan\gamma \approx \gamma$, $\sin\gamma \approx \gamma$, and $\cos\gamma \approx 1$.

EXAMPLE 2.5

Two bars are connected to a roller that slides in a slot, as shown in Figure 2.13. Determine the strains in bar *AP* by: (a) Finding the deformed length of *AP* without small-strain approximation. (b) Using Equation (2.6). (c) Using Equation (2.7).

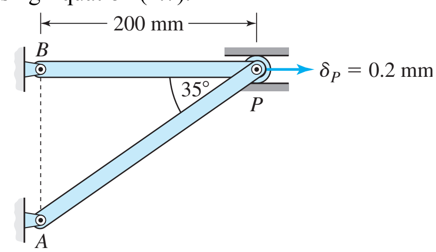


Figure 2.13 Small-strain calculations.

PLAN

(a) An exaggerated deformed shape of the two bars can be drawn and the deformed length of bar *AP* found using geometry. (b) The deformation of bar *AP* can be found by dropping a perpendicular from the final position of point *P* onto the original direction of bar *AP* and using geometry. (c) The deformation of bar *AP* can be found by taking the dot product of the unit vector in the direction of *AP* and the displacement vector of point *P*.

SOLUTION

The length *AP* used in all three methods can be found as $AP = (200\text{ mm})/\cos 35^\circ = 244.155$ mm.

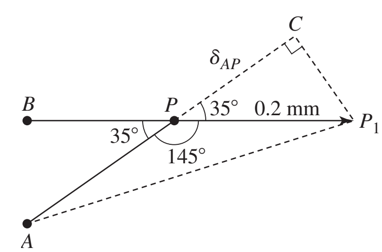


Figure 2.14 Exaggerated deformed shape.

(a) Let point *P* move to point P_1, as shown in Figure 2.14. The angle APP_1 is 145°. From the triangle APP_1 we can find the length AP_1 using the cosine formula and find the strain using Equation (2.1).

$$AP_1 = \sqrt{AP^2 + PP_1^2 - 2(AP)(PP_1)\cos 145^\circ} = 244.3188 \text{ mm} \tag{E1}$$

$$\varepsilon_{AP} = \frac{AP_1 - AP}{AP} = \frac{244.3188\text{ mm} - 244.155\text{ mm}}{244.155\text{ mm}} = 0.67112(10^{-3})\text{ mm/mm} \tag{E2}$$

ANS. $\varepsilon_{AP} = 671.12\ \mu\text{mm/mm}$

(b) We drop a perpendicular from P_1 onto the line in direction of *AP* as shown in Figure 2.14. By the small-strain approximation, the strain in AP is then

$$\delta_{AP} = 0.2\cos 35^\circ = 0.1638\text{ mm} \tag{E3}$$

$$\varepsilon_{AP} = \frac{\delta_{AP}}{AP} = \frac{0.1638\text{ mm}}{244.155\text{ mm}} = 0.67101(10^{-3})\text{ mm/mm} \tag{E4}$$

ANS. $\varepsilon_{AP} = 671.01\,\mu\text{mm/mm}$

(c) Let the unit vectors in the x and y directions be given by $\bar{\mathbf{i}}$ and $\bar{\mathbf{j}}$. The unit vector in direction of AP and the deformation vector $\bar{\mathbf{D}}$ can be written as

$$\bar{\mathbf{i}}_{AP} = \cos 35°\bar{\mathbf{i}} + \sin 35°\bar{\mathbf{j}}, \qquad \bar{\mathbf{D}} = 0.2\bar{\mathbf{i}}, \tag{E5}$$

The strain in AP can be found using Equation (2.7):

$$\delta_{AP} = \bar{\mathbf{D}} \cdot \bar{\mathbf{i}}_{AP} = (0.2\text{ mm})\cos 35 = 0.1638\text{ mm} \tag{E6}$$

$$\varepsilon_{AP} = \frac{\delta_{AP}}{AP} = \frac{0.1638\text{ mm}}{244.155\text{ mm}} = 0.67101(10^{-3})\ \text{mm/mm} \tag{E7}$$

ANS. $\varepsilon_{AP} = 671.01\ \mu\text{mm/mm}$

COMMENTS

1. The calculations for parts (b) and (c) are identical, since there is no difference in the approximation between the two approaches. The strain value for part (a) differs from that in parts (b) and (c) by 0.016%, which is insignificant in engineering calculations.
2. To a small-strain approximation the final length AP_1 is being approximated by length AC.
3. If we do not carry many significant figures in part (a) we may get a prediction of zero strain as the first three significant figures subtract out.

EXAMPLE 2.6

A gap of 0.18 mm exists between the rigid plate and bar B before the load P is applied on the system shown in Figure 2.15. After load P is applied, the axial strain in rod B is –2500 μm/m. Determine the axial strain in rods A.

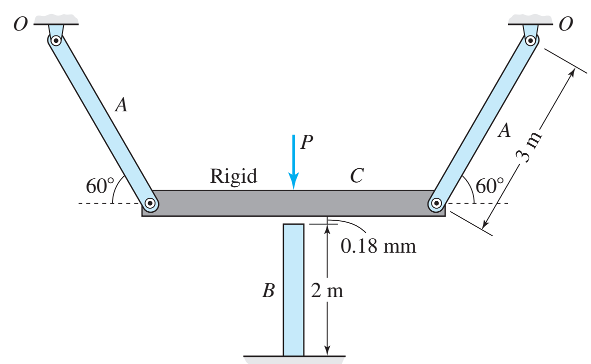


Figure 2.15 Undeformed geometry in Example 2.6.

PLAN

The deformation of bar B can be found from the given strain and related to the displacement of the rigid plate by drawing an approximate deformed shape. We can then relate the displacement of the rigid plate to the deformation of bar A using small-strain approximation.

SOLUTION

From the given strain of bar B we can find the deformation of bar B:

$$\delta_B = \varepsilon_B L_B = (2500)(10^{-6})(2\text{ m}) = 0.005\ \text{ m contraction} \tag{E1}$$

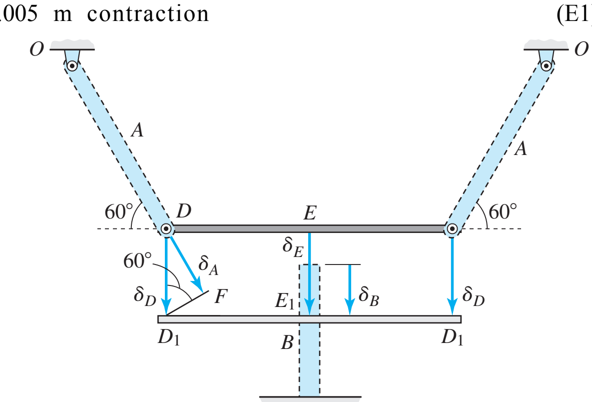


Figure 2.16 Deformed geometry.

Let points D and E be points on the rigid plate. Let the position of these points be D_1 and E_1 after the load P has been applied, as shown in Figure 2.16.

From Figure 2.16 the displacement of point E is

$$\delta_E = \delta_B + 0.00018 \text{ m} = 0.00518 \text{ m} \tag{E2}$$

As the rigid plate moves downward horizontally without rotation, the displacements of points D and E are the same:

$$\delta_D = \delta_E = 0.00518 \text{ m} \tag{E3}$$

We can drop a perpendicular from D_1 to the line in the original direction OD and relate the deformation of bar A to the displacement of point D:

$$\delta_A = \delta_D \sin 60° = (0.00518 \text{ m}) \sin 60° = 0.004486 \text{ m} \tag{E4}$$

The normal strain in A is then

$$\varepsilon_A = \frac{\delta_A}{L_A} = \frac{0.004486 \text{ m}}{3 \text{ m}} = 1.49539(10^{-3}) \text{ m/m} \tag{E5}$$

ANS. $\varepsilon_A = 1495 \ \mu \text{ m/m}$

COMMENTS

1. Equation (E3) is the relationship of points on the rigid bar, whereas Equations (E2) and (E4) are the relationship between the movement of points on the rigid bar and the deformation of the bar. This two-step process simplifies deformation analysis as it reduces the possibility of mistakes in the calculations.
2. We dropped the perpendicular from D_1 to OD and not from D to OD_1 because OD is the original direction, and not OD_1.

EXAMPLE 2.7

Two bars of hard rubber are attached to a rigid disk of radius 20 mm as shown in Figure 2.17. The rotation of the rigid disk by an angle $\Delta\phi$ causes a shear strain at point A of 2000 μrad. Determine the rotation $\Delta\phi$ and the shear strain at point C.

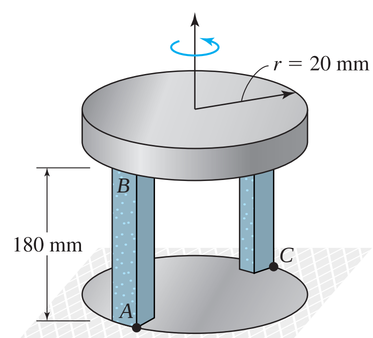


Figure 2.17 Geometry in Example 2.7.

PLAN

The displacement of point B can be related to shear strain at point A as in Example 2.2. All radial lines rotate by equal amounts of $\Delta\phi$ on the rigid disk. We can find $\Delta\phi$ by relating displacement of point B to $\Delta\phi$ assuming small strains. We repeat the calculation for the bar at C to find the strain at C.

SOLUTION

The shear strain at A is $\gamma_A = 2000 \ \mu\text{rad} = 0.002 \text{ rad}$. We draw the approximated deformed shape of the two bars as shown in Figure 2.18a. The displacement of point B is approximately equal to the arc length BB_1, which is related to the rotation of the disk, as shown in Figure 2.18a and b and given as

$$\Delta u_B = (20\text{ mm})(\Delta\phi) \tag{E1}$$

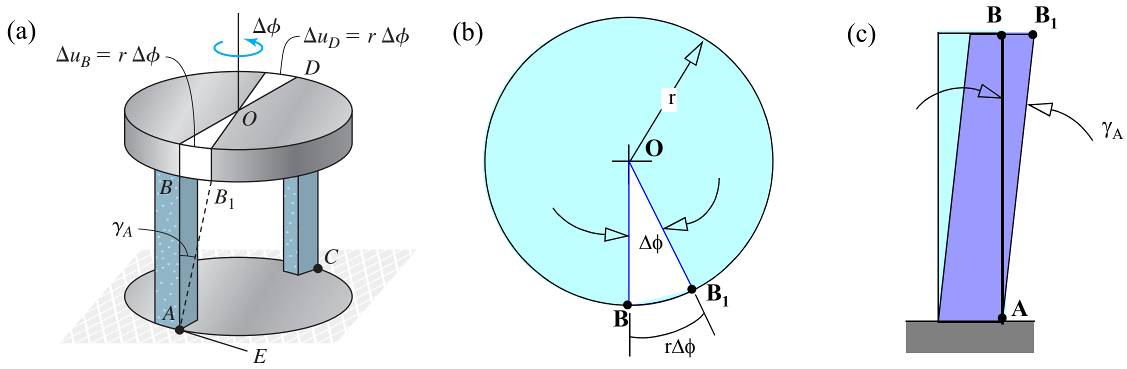


Figure 2.18 (a) Deformed geometry in Example 2.7. (b) Top view of disc. (c) Side view of bar.

The displacement of point B can also be related to the shear strain at A, and we can find $\Delta\phi$ as

$$\tan\gamma_A \approx \gamma_A = \frac{BB_1}{AB} = \frac{\Delta u_B}{AB} = \frac{(20\Delta\phi)\text{ mm}}{180\text{ mm}} = \frac{\Delta\phi}{9} \tag{E2}$$

$$\Delta\phi = 9\gamma_A = (9)(0.002) = 0.018\text{ rad} \tag{E3}$$

ANS. $\Delta\phi = 0.018\ \text{rad}$

The displacement of point D can be found and the shear strain at C obtained from

$$\gamma_C = \frac{\Delta u_D}{CD} = \frac{20\Delta\phi}{180} = \frac{(20\text{ mm})(0.018\text{ rad})}{180\text{ mm}} = 0.002\text{ rad} \tag{E4}$$

ANS. $\gamma_C = 2000\ \mu\text{rad}$

COMMENTS

1. We approximated the arc BB_1 by a straight line, which is valid only if the deformations are small.
2. The shear strain was found from the change in angle formed by the tangent line AE and the axial line AB.
3. In Chapter 5, on the torsion of circular shafts, we will consider a shaft made up of bars and calculate the shear strain due to torsion as in this example.

2.4.1 Vector Approach to Small-Strain Approximation

To calculate strains from known displacements of the pins in truss problems is difficult using the small-strain approximation given by Equation (2.6). Similar algebraic difficulties are encountered in three-dimension. A vector approach helps address these difficulties.

The deformation of the bar in Equation (2.6) is given by $\delta = D\cos\theta$ and can be written in vector form using the dot product:

$$\boxed{\delta = \bar{\mathbf{D}}_{AP}\cdot\bar{\mathbf{i}}_{AP}} \tag{2.7}$$

where $\bar{\mathbf{D}}_{AP}$ is the deformation vector of the bar AP and $\bar{\mathbf{i}}_{AP}$ is the unit vector in the original direction of bar AP. With point A fixed in Figure 2.12 the vector $\bar{\mathbf{D}}_{AP}$ is also the displacement vector of point P. If point A is also displaced, then the deformation vector is obtained by taking the difference between the displacement vectors of point P and point A. If points A and P have coordinates (x_A, y_A, z_A) and (x_P, y_P, z_P), respectively, and are displaced by amounts (u_A, v_A, w_A) and (u_P, v_P, w_P) in the x, y, and z directions, respectively, then the deformation vector $\bar{\mathbf{D}}_{AP}$ and the unit vector $\bar{\mathbf{i}}_{AP}$ can be written as

$$\begin{gathered}\bar{\mathbf{D}}_{AP} = (u_P - u_A)\bar{\mathbf{i}} + (v_P - v_A)\bar{\mathbf{j}} + (w_P - w_A)\bar{\mathbf{k}}\\ \bar{\mathbf{i}}_{AP} = (x_P - x_A)\bar{\mathbf{i}} + (y_P - y_A)\bar{\mathbf{j}} + (z_P - z_A)\bar{\mathbf{k}}\end{gathered} \tag{2.8}$$

where $\bar{\mathbf{i}}$, $\bar{\mathbf{j}}$, and $\bar{\mathbf{k}}$ are the unit vectors in the x, y, and z directions, respectively. The important point to remember about the calculation of $\bar{\mathbf{D}}_{AP}$ and $\bar{\mathbf{i}}_{AP}$ is that the same reference point (A) must be used in calculating deformation vector and the unit vector.

EXAMPLE 2.8*

The displacements of pins of the truss shown Figure 2.19 were computed by the finite-element method (see Section 4.8) and are given below. u and v are the pin displacement x and y directions, respectively. Determine the axial strains in members BC, HB, HC, and HG.

$$u_B = 2.700 \text{ mm} \qquad v_B = -9.025 \text{ mm}$$
$$u_C = 5.400 \text{ mm} \qquad v_C = -14.000 \text{ mm}$$
$$u_G = 8.000 \text{ mm} \qquad v_G = -14.000 \text{ mm}$$
$$u_H = 9.200 \text{ mm} \qquad v_H = -9.025 \text{ mm}$$

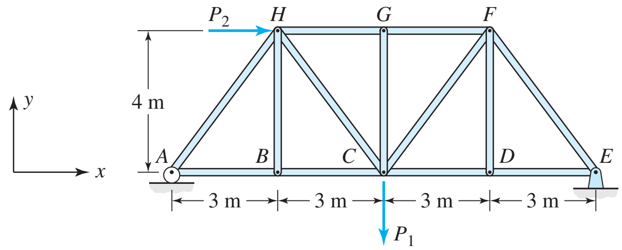


Figure 2.19 Truss in Example 2.8.

PLAN

The deformation vectors for each bar can be found from the given displacements. The unit vectors in directions of the bars *BC, HB, HC,* and *HG* can be determined. The deformation of each bar can be found using Equation (2.7) from which we can find the strains.

SOLUTION

Let the unit vectors in the x and y directions be given by $\bar{\mathbf{i}}$ and $\bar{\mathbf{j}}$, respectively. The deformation vectors for each bar can be found for the given displacement as

$$\bar{\mathbf{D}}_{BC} = (u_C - u_B)\bar{\mathbf{i}} + (v_C - v_B)\bar{\mathbf{j}} = (2.7\bar{\mathbf{i}} - 4.975\bar{\mathbf{j}})\text{ mm} \qquad \bar{\mathbf{D}}_{HB} = (u_B - u_H)\bar{\mathbf{i}} + (v_B - v_H)\bar{\mathbf{j}} = (-6.5\bar{\mathbf{i}})\text{ mm}$$
$$\bar{\mathbf{D}}_{HC} = (u_C - u_H)\bar{\mathbf{i}} + (v_C - v_H)\bar{\mathbf{j}} = (-3.8\bar{\mathbf{i}} - 4.975\bar{\mathbf{j}})\text{ mm} \qquad \bar{\mathbf{D}}_{HG} = (u_G - u_H)\bar{\mathbf{i}} + (v_G - v_H)\bar{\mathbf{j}} = (-1.2\bar{\mathbf{i}} - 4.975\bar{\mathbf{j}})\text{ mm} \tag{E1}$$

The unit vectors in the directions of bars BC, HB, and HG can be found by inspection as these bars are horizontal or vertical:

$$\bar{\mathbf{i}}_{BC} = \bar{\mathbf{i}} \qquad \bar{\mathbf{i}}_{HB} = -\bar{\mathbf{j}} \qquad \bar{\mathbf{i}}_{HG} = \bar{\mathbf{i}} \tag{E2}$$

The position vector from point H to C is $\overline{\mathbf{HC}} = 3\bar{\mathbf{i}} - 4\bar{\mathbf{j}}$. Dividing the position vector by its magnitude we obtain the unit vector in the direction of bar HC:

$$\bar{\mathbf{i}}_{HC} = \frac{\overline{\mathbf{HC}}}{|\overline{\mathbf{HC}}|} = \frac{(3\text{ mm})\bar{\mathbf{i}} - (4\text{ mm})\bar{\mathbf{j}}}{\sqrt{(3\text{ mm})^2 + (4\text{ mm})^2}} = 0.6\bar{\mathbf{i}} - 0.8\bar{\mathbf{j}} \tag{E3}$$

We can find the deformation of each bar from Equation (2.7):

$$\delta_{BC} = \bar{\mathbf{D}}_{BC} \cdot \bar{\mathbf{i}}_{BC} = 2.7\text{ mm} \qquad \delta_{HG} = \bar{\mathbf{D}}_{HG} \cdot \bar{\mathbf{i}}_{HG} = -1.2\text{ mm}$$
$$\delta_{HB} = \bar{\mathbf{D}}_{HB} \cdot \bar{\mathbf{i}}_{HB} = 0 \qquad \delta_{HC} = \bar{\mathbf{D}}_{HC} \cdot \bar{\mathbf{i}}_{HC} = (0.6\text{ mm})(-3.8) + (-4.975\text{ mm})(-0.8) = 1.7\text{ mm} \tag{E4}$$

Finally, Equation (2.2) gives the strains in each bar:

$$\varepsilon_{BC} = \frac{\delta_{BC}}{L_{BC}} = \frac{2.7\text{ mm}}{3 \times 10^3\text{ mm}} = 0.9 \times 10^{-3}\text{ mm/mm} \qquad \varepsilon_{HG} = \frac{\delta_{HG}}{L_{HG}} = \frac{-1.2\text{ mm}}{3 \times 10^3\text{ mm}} = -0.4 \times 10^{-3}\text{ mm/mm}$$
$$\varepsilon_{HB} = \frac{\delta_{HB}}{L_{HB}} = 0 \qquad \varepsilon_{HC} = \frac{\delta_{HC}}{L_{HC}} = \frac{1.7\text{ mm}}{3 \times 10^3\text{ mm}} = 0.340 \times 10^{-3}\text{ mm/mm} \tag{E5}$$

ANS. $\varepsilon_{BC} = 900\ \mu\text{mm/mm} \qquad \varepsilon_{HG} = -400\ \mu\text{mm/mm} \qquad \varepsilon_{HB} = 0 \qquad \varepsilon_{HC} = 340\ \mu\text{mm/mm}$

COMMENTS

1. The zero strain in *HB* is not surprising. By looking at joint *B*, we can see that *HB* is a zero-force member. Though we have yet to establish the relationship between internal forces and deformation, we know intuitively that internal forces will develop if a body deforms.
2. We took a very procedural approach in solving the problem and, as a consequence, did several additional computations. For horizontal bars *BC* and *HG* we could have found the deformation by simply subtracting the u components, and for the vertical bar *HB* we can find the deformation by subtracting the v component. But care must be exercised in determining whether the bar is in extension or in contraction, for otherwise an error in sign can occur.
3. In Figure 2.20 point H is held fixed (reference point), and an exaggerated relative movement of point C is shown by the vector $\bar{\mathbf{D}}_{HC}$. The calculation of the deformation of bar *HC* is shown graphically.

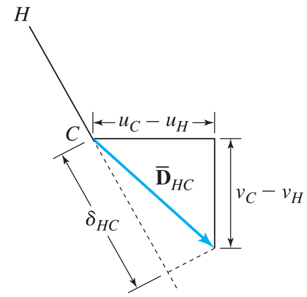


Figure 2.20 Visualization of the deformation vector for bar *HC*.

4. Suppose that instead of finding the relative movement of point *C* with respect to *H*, we had used point *C* as our reference point and found the relative movement of point *H*. The deformation vector would be $\overline{\mathbf{D}}_{CH}$, which is equal to $-\overline{\mathbf{D}}_{HC}$. But the unit vector direction would also reverse, that is, we would use $\bar{\mathbf{i}}_{CH}$, which is equal to $-\bar{\mathbf{i}}_{HC}$. Thus the dot product to find the deformation would yield the same number and the same sign. The result independent of the reference point is true only for small strains, which we have implicitly assumed.

PROBLEM SET 2.1

Average normal strains

2.1 An 80-cm stretch cord is used to tie the rear of a canoe to the car hook, as shown in Figure P2.1. In the stretched position the cord forms the side *AB* of the triangle shown. Determine the average normal strain in the stretch cord.

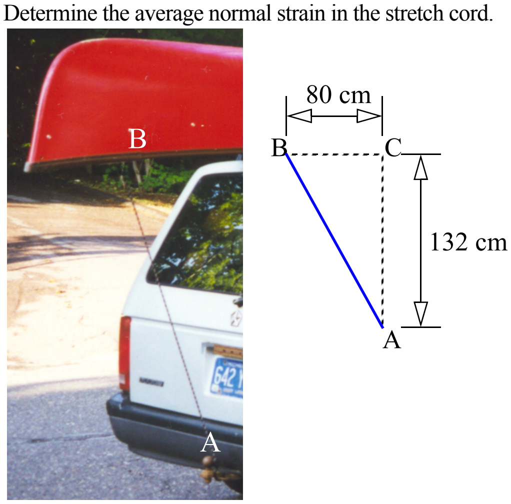


Figure P2.1

2.2 The diameter of a spherical balloon shown in Figure P2.2 changes from 250 mm to 252 mm. Determine the change in the average circumferential normal strain.

Figure P2.2

2.3 Two rubber bands are used for packing an air mattress for camping as shown in Figure P2.3. The undeformed length of a rubber band is 7 in. Determine the average normal strain in the rubber bands if the diameter of the mattress is 4.1 in. at the section where the rubber bands are on the mattress.

Figure P2.3

2.4 A canoe on top of a car is tied down using rubber stretch cords, as shown in Figure P2.4*a*. The undeformed length of the stretch cord is 40 in. Determine the average normal strain in the stretch cord assuming that the path of the stretch cord over the canoe can be approximated as shown in Figure P2.4*b*.

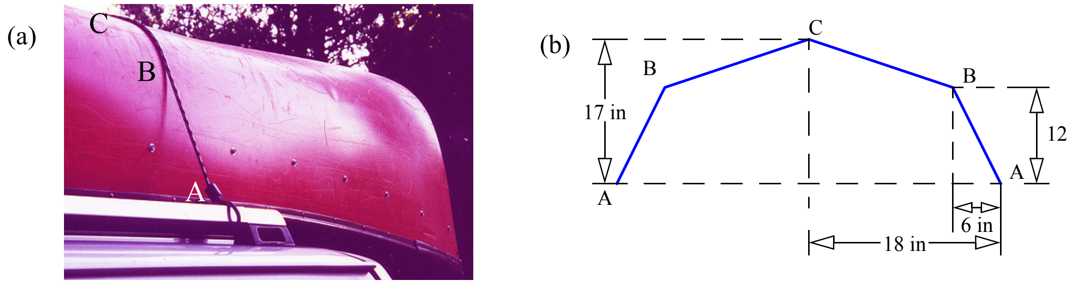


Figure P2.4

2.5 The cable between two poles shown in Figure P2.5 is taut before the two traffic lights are hung on it. The lights are placed symmetrically at 1/3 the distance between the poles. Due to the weight of the traffic lights the cable sags as shown. Determine the average normal strain in the cable.

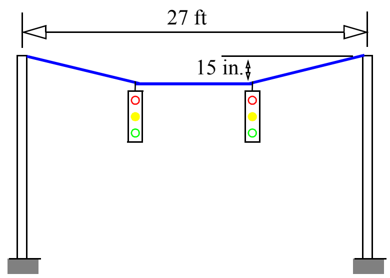


Figure P2.5

2.6 The displacements of the rigid plates in x direction due to the application of the forces in Figure P2.6 are $u_B = -1.8$ mm, $u_C = 0.7$ mm, and $u_D = 3.7$ mm. Determine the axial strains in the rods in sections *AB*, *BC*, and *CD*.

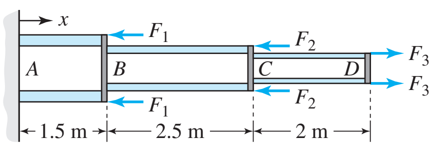


Figure P2.6

2.7 The average normal strains in the bars due to the application of the forces in Figure P2.6 are $\varepsilon_{AB} = -800\ \mu$, $\varepsilon_{BC} = 600\ \mu$, and $\varepsilon_{CD} = 1100\ \mu$. Determine the movement of point D with respect to the left wall.

2.8 Due to the application of the forces, the rigid plate in Figure P2.8 moves 0.0236 in to the right. Determine the average normal strains in bars A and B.

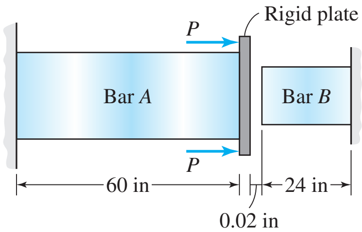


Figure P2.8

2.9 The average normal strain in bar A due to the application of the forces in Figure P2.8, was found to be 2500 μ in./in. Determine the normal strain in bar B.

2.10 The average normal strain in bar B due to the application of the forces in Figure P2.8 was found to be -4000 μ in./in. Determine the normal strain in bar A.

2.11 Due to the application of force P, point B in Figure P2.11 moves upward by 0.06 in. If the length of bar A is 24 in., determine the average normal strain in bar A.

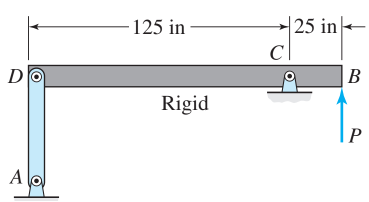


Figure P2.11

2.12 The average normal strain in bar *A* due to the application of force *P* in Figure P2.11 was found to be –6000 μ in./in. If the length of bar *A* is 36 in., determine the movement of point *B*.

2.13 Due to the application of force *P*, point *B* in Figure P2.13 moves upward by 0.06 in. If the length of bar *A* is 24 in., determine the average normal strain in bar *A*.

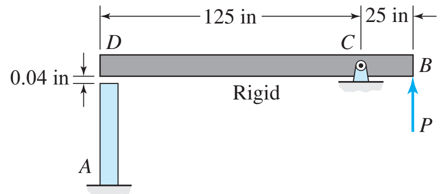


Figure P2.13

2.14 The average normal strain in bar *A* due to the application of force *P* in Figure P2.13 was found to be –6000 μ in./in. If the length of bar *A* is 36 in., determine the movement of point *B*.

2.15 Due to the application of force *P*, point *B* in Figure P2.15 moves upward by 0.06 in. If the lengths of bars *A* and *F* are 24 in., determine the average normal strain in bars *A* and *F*.

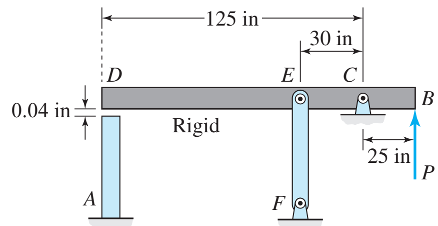


Figure P2.15

2.16 The average normal strain in bar *A* due to the application of force *P* in Figure P2.15 was found to be –5000 μ in./in. If the lengths of bars *A* and *F* are 36 in., determine the movement of point *B* and the average normal strain in bar *F*.

2.17 The average normal strain in bar *F* due to the application of force *P*, in Figure P2.15 was found to be -2000 μ in./in. If the lengths of bars *A* and *F* are 36 in., determine the movement of point *B* and the average normal strain in bar *A*.

2.18 Due to the application of force *P*, point *B* in Figure P2.18 moves left by 0.75 mm. If the length of bar *A* is 1.2 m, determine the average normal strain in bar *A*.

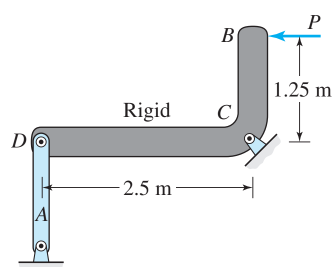


Figure P2.18

2.19 The average normal strain in bar *A* due to the application of force *P* in Figure P2.18 was found to be –2000 μ m /m. If the length of bar *A* is 2 m, determine the movement of point *B*.

2.20 Due to the application of force *P*, point *B* in Figure P2.20 moves left by 0.75 mm. If the length of bar *A* is 1.2 m, determine the average normal strain in bar *A*.

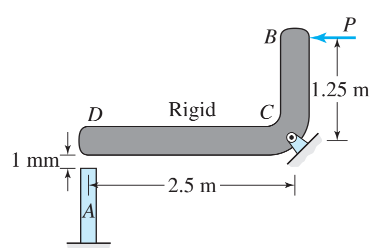


Figure P2.20

2.21 The average normal strain in bar *A* due to the application of force *P* in Figure P2.20 was found to be –2000 μ m/m. If the length of bar *A* is 2 m, determine the movement of point *B*.

2.22 Due to the application of force *P*, point *B* in Figure P2.22 moves left by 0.75 mm. If the lengths of bars *A* and *F* are 1.2 m, determine the average normal strains in bars *A* and *F*.

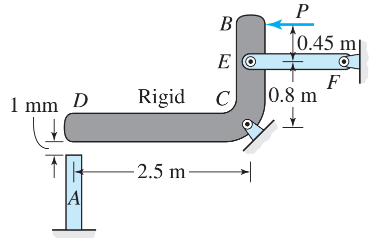


Figure P2.22

2.23 The average normal strain in bar *A* due to the application of force *P* in Figure P2.22 was found to be –2500 μ m/m. Bars *A* and *F* are 2 m long. Determine the movement of point *B* and the average normal strain in bar *F*.

2.24 The average normal strain in bar *F* due to the application of force *P* in Figure P2.22 was found to be 1000 μ m/m. Bars *A* and *F* are 2 m long. Determine the movement of point *B* and the average normal strain in bar *A*.

2.25 Two bars of equal lengths of 400 mm are welded to rigid plates at right angles. The right angles between the bars and the plates are preserved as the rigid plates are rotated by an angle of ψ as shown in Figure P2.25. The distance between the bars is $h = 50$ mm. The average normal strains in bars AB and CD were determined as -2500 μ mm/mm and 3500 μ mm/mm, respectively. Determine the radius of curvature R and the angle ψ.

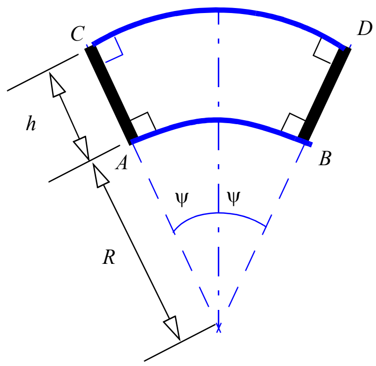


Figure P2.25

2.26 Two bars of equal lengths of 30 in. are welded to rigid plates at right angles. The right angles between the bars and the plates are preserved as the rigid plates are rotated by an angle of $\psi = 1.25^{o}$ as shown in Figure P2.25. The distance between the bars is h = 2 in. If the average normal strain in bar *AB* is -1500 μ in./in., determine the strain in bar *CD*.

2.27 Two bars of equal lengths of 48 in. are welded to rigid plates at right angles. The right angles between the bars and the plates are preserved as the rigid plates are rotated by an angle of ψ as shown in Figure P2.27. The average normal strains in bars *AB* and *CD* were determined as -2000 μ in./in. and 1500 μ in./in., respectively. Determine the location h of a third bar *EF* that should be placed such that it has zero normal strain.

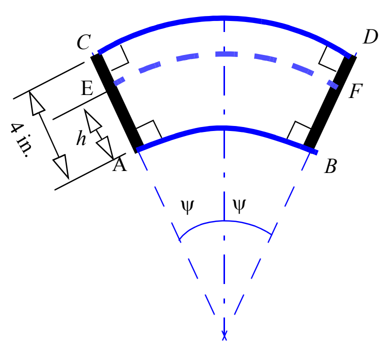


Figure P2.27

Average shear strains

2.28 A rectangular plastic plate deforms into a shaded shape, as shown in Figure P2.28. Determine the average shear strain at point A.

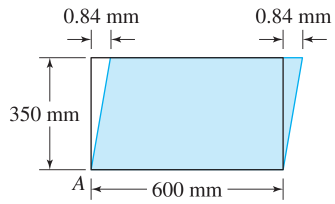


Figure P2.28

2.29 A rectangular plastic plate deforms into a shaded shape, as shown in Figure P2.29. Determine the average shear strain at point A.

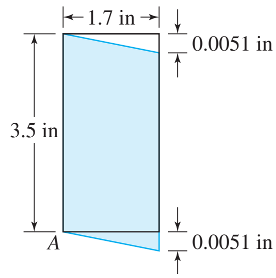


Figure P2.29

2.30 A rectangular plastic plate deforms into a shaded shape, as shown in Figure P2.30. Determine the average shear strain at point A.

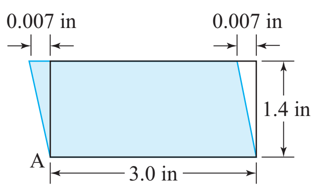


Figure P2.30

2.31 A rectangular plastic plate deforms into a shaded shape, as shown in Figure P2.31. Determine the average shear strain at point A.

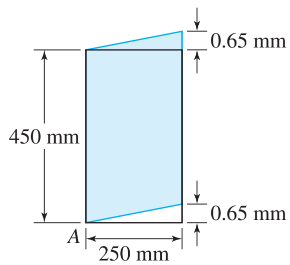


Figure P2.31

2.32 A rectangular plastic plate deforms into a shaded shape, as shown in Figure P2.32. Determine the average shear strain at point A.

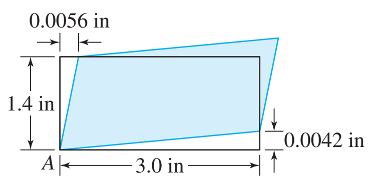


Figure P2.32

2.33 A rectangular plastic plate deforms into a shaded shape, as shown in Figure P2.33. Determine the average shear strain at point A.

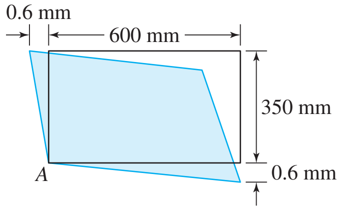


Figure P2.33

2.34 A thin triangular plate ABC forms a right angle at point A, as shown in Figure P2.34. During deformation, point A moves vertically down by $\delta_A = 0.005$ in. Determine the average shear strains at point A.

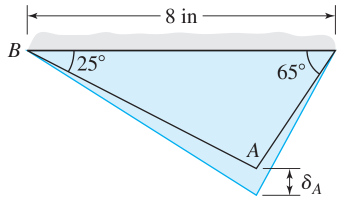


Figure P2.34

2.35 A thin triangular plate ABC forms a right angle at point A, as shown in Figure P2.35. During deformation, point A moves vertically down by $\delta_A = 0.006$ in. Determine the average shear strains at point A.

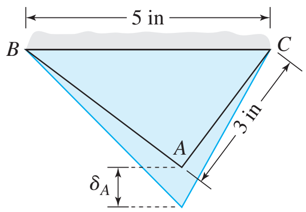


Figure P2.35

2.36 A thin triangular plate ABC forms a right angle at point A, as shown in Figure P2.36. During deformation, point A moves vertically down by $\delta_A = 0.75$ mm. Determine the average shear strains at point A.

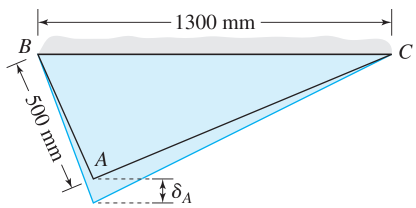


Figure P2.36

2.37 A thin triangular plate *ABC* forms a right angle at point *A*. During deformation, point A moves horizontally by δ_A =0.005 in., as shown in Figure P2.37. Determine the average shear strains at point *A*.

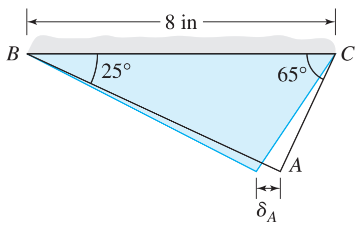


Figure P2.37

2.38 A thin triangular plate *ABC* forms a right angle at point *A*. During deformation, point *A* moves horizontally by δ_A =0.008 in., as shown in Figure P2.38. Determine the average shear strains at point *A*.

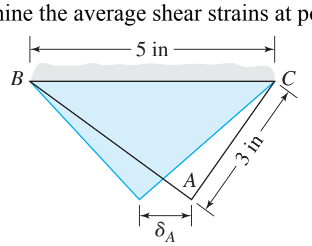


Figure P2.38

2.39 A thin triangular plate *ABC* forms a right angle at point *A*. During deformation, point *A* moves horizontally by δ_A =0.90 mm, as shown in Figure P2.39. Determine the average shear strains at point *A*.

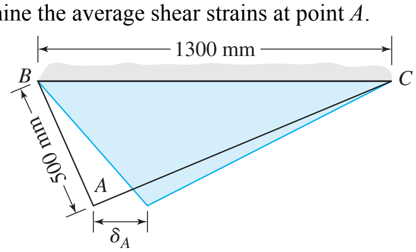


Figure P2.39

2.40 Bar *AB* is bolted to a plate along the diagonal as shown in Figure P2.40. The plate experiences an average strain in the *x* direction $\varepsilon = 500\ \mu\text{in./in.}$. Determine the average normal strain in the bar *AB*.

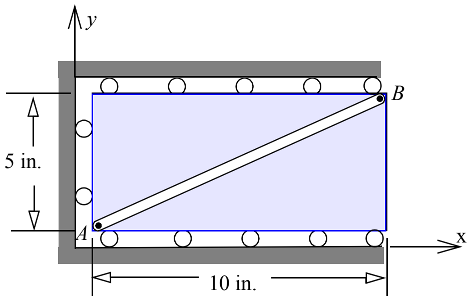


Figure P2.40

2.41 Bar *AB* is bolted to a plate along the diagonal as shown in Figure P2.40. The plate experiences an average strain in the *y* direction $\varepsilon = -1200\ \mu\text{mm/mm}$. Determine the average normal strain in the bar *AB*.

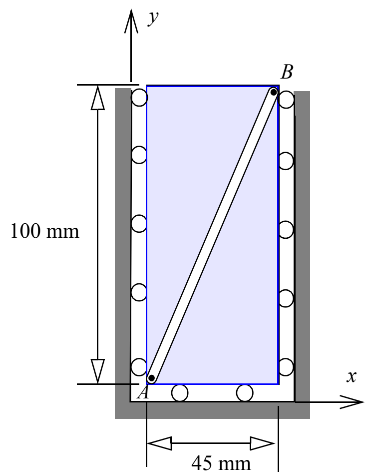


Figure P2.41

2.42 A right angle bar *ABC* is welded to a plate as shown in Figure P2.42. Points *B* are fixed. The plate experiences an average strain in the *x* direction $\varepsilon = -1000\mu\text{mm/mm}$. Determine the average normal strain in *AB*.

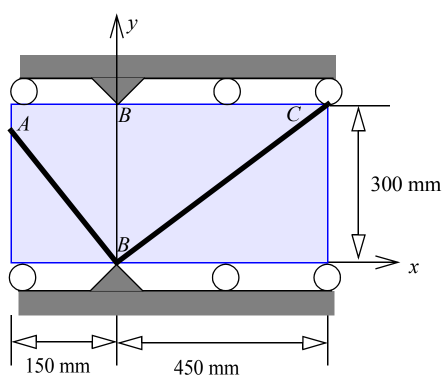


Figure P2.42

2.43 A right angle bar *ABC* is welded to a plate as shown in Figure P2.42. Points *B* are fixed. The plate experiences an average strain in the *x* direction $\varepsilon = 700\ \mu\text{mm/mm}$. Determine the average normal strain in *BC*.

2.44 A right angle bar *ABC* is welded to a plate as shown in Figure P2.42. Points *B* are fixed. The plate experiences an average strain in the *x* direction $\varepsilon = -800\ \mu\text{mm/mm}$. Determine the average shear strain at point *B* in the bar.

2.45 A right angle bar *ABC* is welded to a plate as shown in Figure P2.45. Points *B* are fixed. The plate experiences an average strain in the *y* direction $\varepsilon = 800\ \mu\text{in./in.}$ Determine the average normal strain in *AB*.

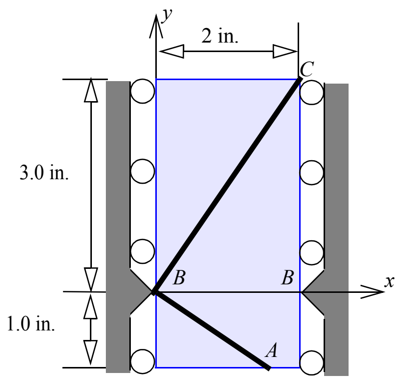


Figure P2.45

2.46 A right angle bar *ABC* is welded to a plate as shown in Figure P2.45. Points *B* are fixed. The plate experiences an average strain in the *y* direction $\varepsilon = -500\ \mu\text{in./in.}$ Determine the average normal strain in *BC*.

2.47 A right angle bar *ABC* is welded to a plate as shown in Figure P2.45. Points *B* are fixed. The plate experiences an average strain in the *y* direction $\varepsilon = 600\ \mu\text{in./in.}$ Determine the average shear strain at *B* in the bar.

2.48 The diagonals of two squares form a right angle at point *A* in Figure P2.48. The two rectangles are pulled horizontally to a deformed shape, shown by colored lines. The displacements of points *A* and *B* are $\delta_A = 0.4$ mm and $\delta_B = 0.8$ mm. Determine the average shear strain at point *A*.

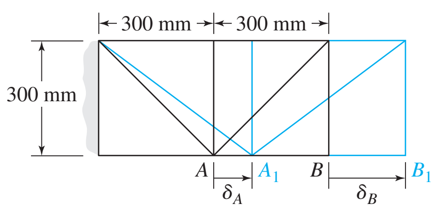


Figure P2.48

2.49 The diagonals of two squares form a right angle at point A in Figure P2.48. The two rectangles are pulled horizontally to a deformed shape, shown by colored lines. The displacements of points A and B are $\delta_A = 0.3$ mm and $\delta_B = 0.9$ mm. Determine the average shear strain at point A $\delta_A = 0.3$ mm and $\delta_B = 0.9$ mm.

Small-strain approximations

2.50 The roller at P slides in the slot by the given amount shown in Figure P2.50. Determine the strains in bar AP by (a) finding the deformed length of AP without the small-strain approximation, (b) using Equation (2.6), and (c) using Equation (2.7).

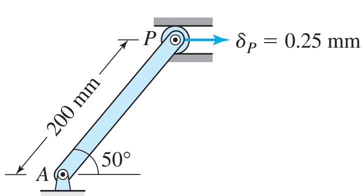


Figure P2.50

2.51 The roller at P slides in the slot by the given amount shown in Figure P2.51. Determine the strains in bar AP by (a) finding the deformed length of AP without small-strain approximation, (b) using Equation (2.6), and (c) using Equation (2.7).

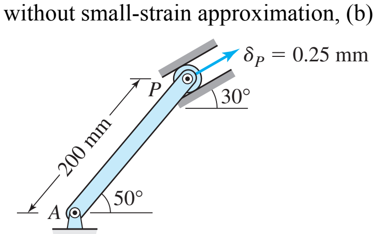


Figure P2.51

2.52 The roller at P slides in a slot by the amount shown in Figure P2.52. Determine the deformation in bars AP and BP using the small-strain approximation.

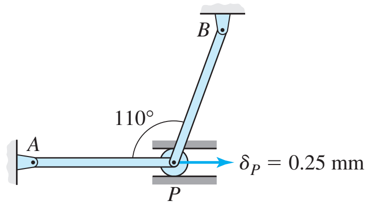


Figure P2.52

2.53 The roller at P slides in a slot by the amount shown in Figure P2.53. Determine the deformation in bars AP and BP using the small-strain approximation.

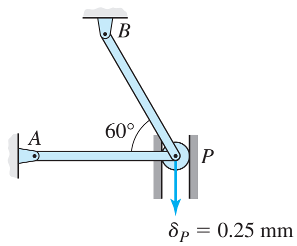


Figure P2.53

2.54 The roller at P slides in a slot by the amount shown in Figure P2.54. Determine the deformation in bars AP and BP using the small-strain approximation.

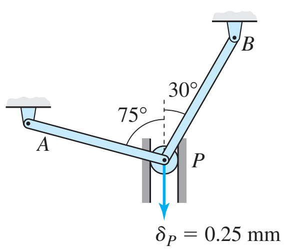


Figure P2.54

2.55 The roller at P slides in a slot by the amount shown in Figure P2.55. Determine the deformation in bars AP and BP using the small-strain approximation.

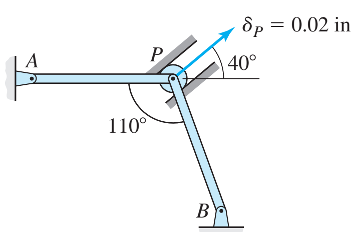


Figure P2.55

2.56 The roller at P slides in a slot by the amount shown in Figure P2.56. Determine the deformation in bars AP and BP using the small-strain approximation.

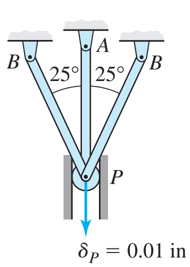


Figure P2.56

2.57 The roller at P slides in a slot by the amount shown in Figure P2.57. Determine the deformation in bars AP and BP using the small-strain approximation.

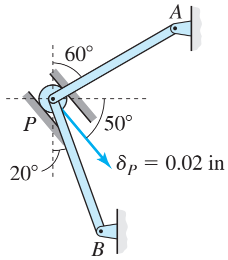


Figure P2.57

2.58 A gap of 0.004 in. exists between the rigid bar and bar *A* before the load *P* is applied in Figure P2.58. The rigid bar is hinged at point *C*. The strain in bar *A* due to force *P* was found to be –600 μ in./in. Determine the strain in bar *B*. The lengths of bars *A* and *B* are 30 in. and 50 in., respectively.

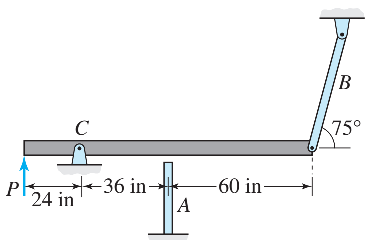


Figure P2.58

2.59 A gap of 0.004 in. exists between the rigid bar and bar *A* before the load *P* is applied in Figure P2.58. The rigid bar is hinged at point *C*. The strain in bar *B* due to force *P* was found to be 1500 μ in./in. Determine the strain in bar *A*. The lengths of bars *A* and *B* are 30 in. and 50 in., respectively.

Vector approach to small-strain approximation

2.60 The pin displacements of the truss in Figure P2.60 were computed by the finite-element method. The displacements in *x* and *y* directions given by *u* and v are given in Table P2.60. Determine the axial strains in members *AB*, *BF*, *FG*, and *GB*.

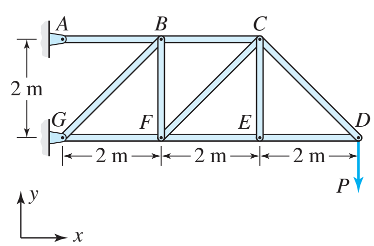


Figure P2.60

TABLE P2.60

u_B = 12.6 mm	v_B = –24.48 mm
u_C = 21.0 mm	v_C = –69.97 mm
u_D = –16.8 mm	v_D = –119.65 mm
u_E = –12.6 mm	v_E = –69.97 mm
u_F = –8.4 mm	v_F = –28.68 mm

2.61 The pin displacements of the truss in Figure P2.60 were computed by the finite-element method. The displacements in *x* and *y* directions given by *u* and v are given in Table P2.60. Determine the axial strains in members *BC*, *CF*, and *FE*.

2.62 The pin displacements of the truss in Figure P2.60 were computed by the finite-element method. The displacements in *x* and *y* directions given by *u* and v are given in Table P2.60. Determine the axial strains in members *ED*, *DC*, and *CE*.

2.63 The pin displacements of the truss in Figure P2.63 were computed by the finite-element method. The displacements in x and y directions given by u and v are given in Table P2.63. Determine the axial strains in members AB, BG, GA, and AH.

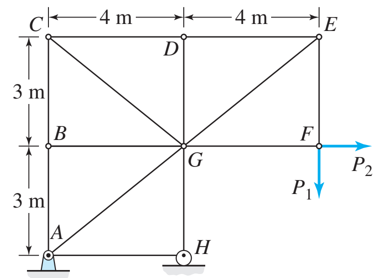


Figure P2.63

TABLE P2.63

u_B = 7.00 mm	v_B = 1.500 mm
u_C = 17.55 mm	v_C = 3.000 mm
u_D = 20.22 mm	v_D = –4.125 mm
u_E = 22.88 mm	v_E = –32.250 mm
u_F = 9.00 mm	v_F = –33.750 mm
u_G = 7.00 mm	v_G = –4.125 mm
u_H = 0	v_H = 0

2.64 The pin displacements of the truss in Figure P2.63 were computed by the finite-element method. The displacements in x and y directions given by u and v are given in Table P2.63.Determine the axial strains in members BC, CG, GB, and CD.

2.65 The pin displacements of the truss in Figure P2.63 were computed by the finite-element method. The displacements in x and y directions given by u and v are given in Table P2.63.Determine the axial strains in members GF, FE, EG, and DE.

2.66 Three poles are pin connected to a ring at P and to the supports on the ground. The ring slides on a vertical rigid pole by 2 in, as shown in Figure P2.66. The coordinates of the four points are as given. Determine the normal strain in each bar due to the movement of the ring.

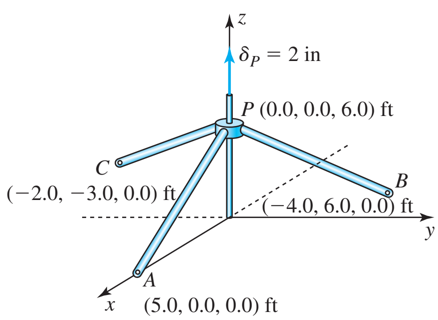


Figure P2.66

MoM in Action: Challenger Disaster

On January 28th, 1986, the space shuttle Challenger (Figure 2.21a) exploded just 73 seconds into the flight, killing seven astronauts. The flight was to have been the first trip for a civilian, the school-teacher Christa McAuliffe. Classrooms across the USA were preparing for the first science class ever taught from space. The explosion shocked millions watching the takeoff and a presidential commission was convened to investigate the cause. Shuttle flights were suspended for nearly two years.

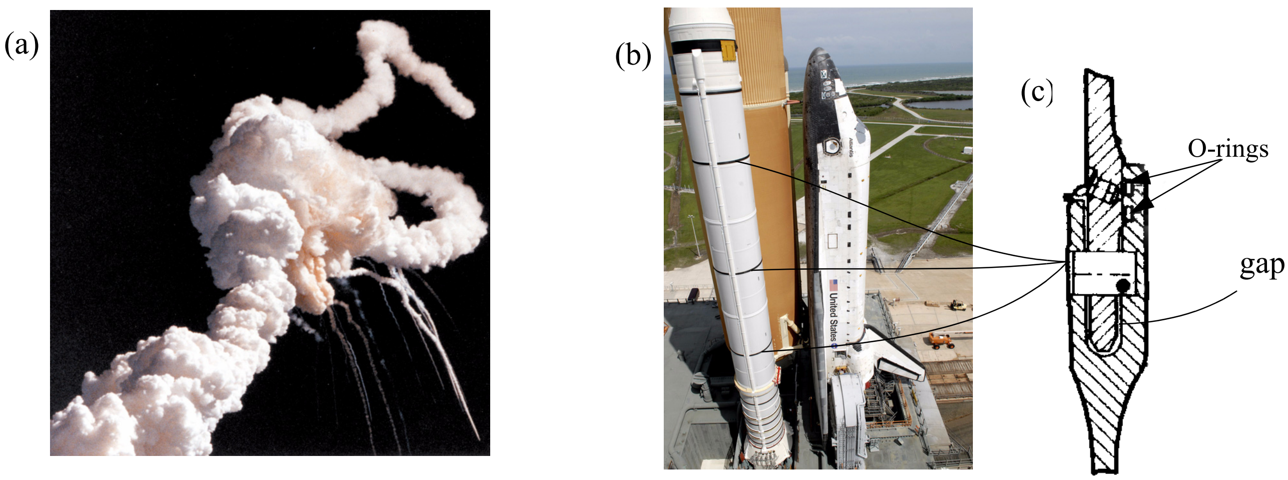


Figure 2.21 (a) Challenger explosion during flight (b) Shuttle Atlantis (c) O-ring joint.

The Presidential commission established that combustible gases from the solid rocket boosters had ignited, causing the explosion. These gases had leaked through the joint between the two lower segments of the boosters on the space shuttle's right side. The boosters of the *Challenger*, like those of the shuttle Atlantis (Figure 2.21b), were assembled using the O-ring joints illustrated in Figure 2.21c. When the gap between the two segments is 0.004 in. or less, the rubber O-rings are in contact with the joining surfaces and there is no chance of leak. At the time of launch, however, the gap was estimated to have exceeded 0.017 in.

But why? Apparently, prior launches had permanently enlarged diameter of the segments at some places, so that they were no longer round. Launch forces caused the segments to move further apart. Furthermore, the O-rings could not return to their uncompressed shape, because the material behavior alters dramatically with temperature. A compressed rubber O-ring at 78° F is five times more responsive in returning to its uncompressed shape than an O-ring at 30° F. The temperature around the joint varied from approximately 28° F on the cold shady side to 50° F in the sun.

Two engineers at Morton Thiokol, a contractor of NASA, had seen gas escape at a previous launch and had recommended against launching the shuttle when the outside air temperature is below 50° F. Thiokol management initially backed their engineer's recommendation but capitulated to desire to please their main customer, NASA. The NASA managers felt under political pressure to establish the space shuttle as a regular, reliable means of conducting scientific and commercial missions in space. Roger Boisjoly, one of the Thiokol engineers was awarded the Prize for Scientific Freedom and Responsibility by American Association for the Advancement of Science for his professional integrity and his belief in engineer's rights and responsibilities.

The accident came about because the deformation at launch was in excess of the design's allowable deformation. An administrative misjudgment of risk assessment and the potential benefits had overruled the engineers.

2.5 STRAIN COMPONENTS

Let u, v, and w be the displacements in the x, y, and z directions, respectively. Figure 2.22 and Equations (2.9a) through (2.9i) define average engineering strain components:

$$\varepsilon_{xx} = \frac{\Delta u}{\Delta x} \tag{2.9a}$$

$$\varepsilon_{yy} = \frac{\Delta \mathrm{v}}{\Delta y} \tag{2.9b}$$

$$\varepsilon_{zz} = \frac{\Delta w}{\Delta z} \tag{2.9c}$$

$$\gamma_{xy} = \frac{\Delta u}{\Delta y} + \frac{\Delta \mathrm{v}}{\Delta x} \tag{2.9d}$$

$$\gamma_{yx} = \frac{\Delta \mathrm{v}}{\Delta x} + \frac{\Delta u}{\Delta y} = \gamma_{xy} \tag{2.9e}$$

$$\gamma_{yz} = \frac{\Delta \mathrm{v}}{\Delta z} + \frac{\Delta w}{\Delta y} \tag{2.9f}$$

$$\gamma_{zy} = \frac{\Delta w}{\Delta y} + \frac{\Delta \mathrm{v}}{\Delta z} = \gamma_{yz} \tag{2.9g}$$

$$\gamma_{zx} = \frac{\Delta w}{\Delta x} + \frac{\Delta u}{\Delta z} \tag{2.9h}$$

$$\gamma_{xz} = \frac{\Delta u}{\Delta z} + \frac{\Delta w}{\Delta x} = \gamma_{zx} \tag{2.9i}$$

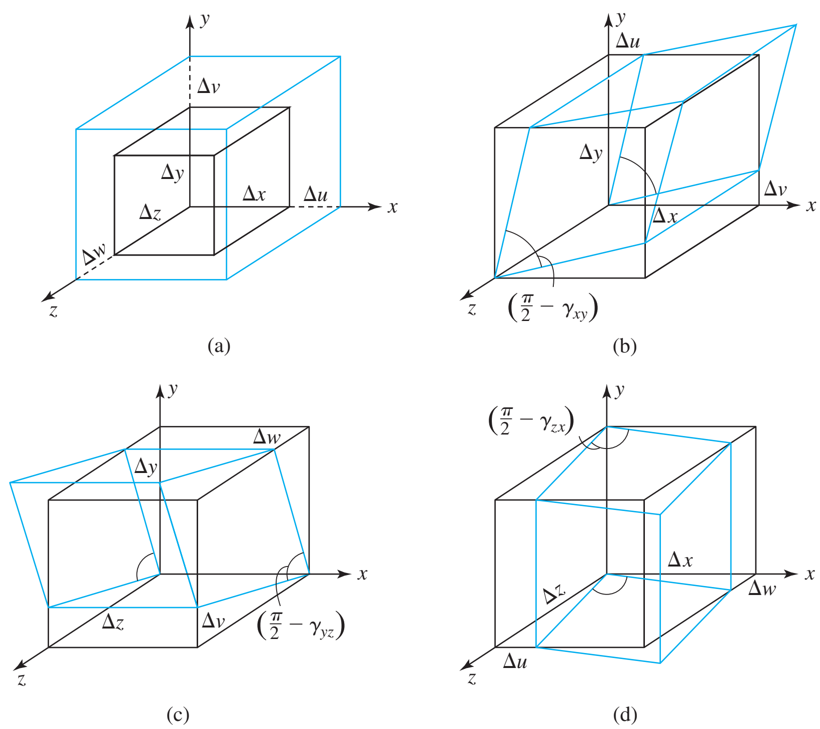


Figure 2.22 (*a*) Normal strains. (*b*) Shear strain γ_{xy}. (*c*) Shear strain γ_{yz}. (*d*) Shear strain γ_{zx}.

Equations (2.9a) through (2.9i) show that strain at a point has nine components in three dimensions, but only six are independent because of the symmetry of shear strain. The symmetry of shear strain makes intuitive sense. The change of angle between the x and y directions is obviously the same as between the y and x directions. In Equations (2.9a) through (2.9i) the first subscript is the direction of displacement and the second the direction of the line element. But because of the symmetry of shear strain, the order of the subscripts is immaterial. Equation (2.10) shows the components as an engineering strain matrix. The matrix is symmetric because of the symmetry of shear strain.

$$\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{2.10}$$

2.5.1 Plane Strain

Plane strain is one of two types of two-dimensional idealizations in mechanics of materials. In Chapter 1 we saw the other type, plane stress. We will see the difference between the two types of idealizations in Chapter 3. By two-dimensional we imply that one of the coordinates does not play a role in the solution of the problem. Choosing z to be that coordinate, we set all strains with subscript z to be zero, as shown in the strain matrix in Equation (2.11). Notice that in plane strain, four components of strain are needed though only three are independent because of the symmetry of shear strain.

$$\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.11}$$

The assumption of plane strain is often made in analyzing very thick bodies, such as points around tunnels, mine shafts in earth, or a point in the middle of a thick cylinder, such as a submarine hull. In thick bodies we can expect a point has to push a lot of material in the thickness direction to move. Hence the strains in the this direction should be small. It is not zero, but it is small enough to be neglected. Plane strain is a mathematical approximation made to simplify analysis.

EXAMPLE 2.9

Displacements u and v in x and y directions, respectively, were measured at many points on a body by the geometric Moiré method (See Section 2.7). The displacements of four points on the body of Figure 2.23 are as given. Determine strains ε_{xx}, ε_{yy}, and γ_{xy} at point A.

$$u_A = -0.0100 \text{ mm} \qquad v_A = 0.0100 \text{ mm}$$
$$u_B = -0.0050 \text{ mm} \qquad v_B = -0.0112 \text{ mm}$$
$$u_C = 0.0050 \text{ mm} \qquad v_C = 0.0068 \text{ mm}$$
$$u_D = 0.0100 \text{ mm} \qquad v_D = 0.0080 \text{ mm}$$

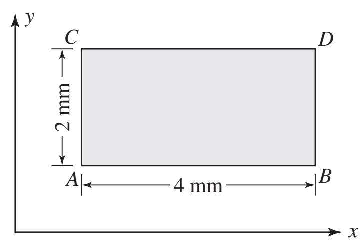


Figure 2.23 Undeformed geometry in Example 2.9.

PLAN

We can use point A as our reference point and calculate the relative movement of points B and C and find the strains from Equations (2.9a), (2.9b), and (2.9d).

SOLUTION

The relative movements of points B and C with respect to A are

$$u_B - u_A = 0.0050 \text{ mm} \qquad v_B - v_A = -0.0212 \text{ mm} \tag{E1}$$

$$u_C - u_A = 0.0150 \text{ mm} \qquad v_C - v_A = -0.0032 \text{ mm} \tag{E2}$$

The normal strains ε_{xx} and ε_{yy} can be calculated as

$$\varepsilon_{xx} = \frac{u_B - u_A}{x_B - x_A} = \frac{0.0050 \text{ mm}}{4 \text{ mm}} = 0.00125 \text{ mm/mm} \tag{E3}$$

$$\varepsilon_{yy} = \frac{v_C - v_A}{y_C - y_A} = \frac{-0.0032 \text{ mm}}{2 \text{ mm}} = -0.0016 \text{ mm/mm} \tag{E4}$$

ANS. $\varepsilon_{xx} = 1250\ \mu\text{mm/mm}$ $\varepsilon_{yy} = -1600\ \mu\text{mm/mm}$

From Equation (2.9d) the shear strain can be found as

$$\gamma_{xy} = \frac{v_B - v_A}{x_B - x_A} + \frac{u_C - u_A}{y_C - y_A} = \frac{-0.0212\text{ mm}}{4\text{ mm}} + \frac{0.0150\text{ mm}}{2\text{ mm}} = 0.0022\text{ rad} \tag{E5}$$

ANS. $\gamma_{xy} = 2200\ \mu\text{rads}$

COMMENT

1. Figure 2.24 shows an exaggerated deformed shape of the rectangle. Point A moves to point A_1; similarly, the other points move to B_1, C_1, and D_1. By drawing the undeformed rectangle from point A, we can show the relative movements of the three points. We could have calculated the length of A_1B from the Pythagorean theorem as $A_1B_1 = \sqrt{(4-0.005)^2 + (-0.0212)^2} = 3.995056\text{ mm}$, which would yield the following strain value:

$$\varepsilon_{xx} = \frac{A_1B_1 - AB}{AB} = 1236\ \mu\text{mm/mm}.$$

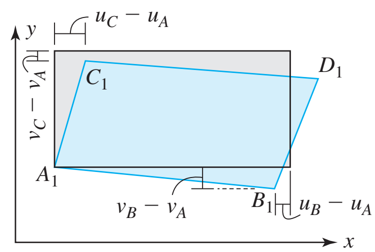


Figure 2.24 Elaboration of comment.

The difference between the two calculations is 1.1%. We will have to perform similar tedious calculations to find the other two strains if we want to gain an additional accuracy of 1% or less. But notice the simplicity of the calculations that come from a small-strain approximation.

2.6 STRAIN AT A POINT

In Section 2.5 the lengths Δx, Δy, and Δz were finite. If we shrink these lengths to zero in Equations (2.9a) through (2.9i), we obtain the definition of strain at a point. Because the limiting operation is in a given direction, we obtain partial derivatives and not the ordinary derivatives:

$$\varepsilon_{xx} = \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x}\right) = \frac{\partial u}{\partial x} \tag{2.12a}$$

$$\varepsilon_{yy} = \lim_{\Delta y \to 0}\left(\frac{\Delta v}{\Delta y}\right) = \frac{\partial v}{\partial y} \tag{2.12b}$$

$$\varepsilon_{zz} = \lim_{\Delta z \to 0}\left(\frac{\Delta w}{\Delta z}\right) = \frac{\partial w}{\partial z} \tag{2.12c}$$

$$\gamma_{xy} = \gamma_{yx} = \lim_{\substack{\Delta x \to 0\\ \Delta y \to 0}}\left(\frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x}\right) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{2.12d}$$

$$\gamma_{yz} = \gamma_{zy} = \lim_{\substack{\Delta y \to 0\\ \Delta z \to 0}}\left(\frac{\Delta v}{\Delta z} + \frac{\Delta w}{\Delta y}\right) = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \tag{2.12e}$$

$$\gamma_{zx} = \gamma_{xz} = \lim_{\substack{\Delta x \to 0\\ \Delta z \to 0}}\left(\frac{\Delta w}{\Delta x} + \frac{\Delta u}{\Delta z}\right) = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \tag{2.12f}$$

Equations (2.12a) through (2.12f) show that engineering strain has two subscripts, indicating both the direction of deformation and the direction of the line element that is being deformed. Thus it would seem that engineering strain is also a second-order tensor. However, unlike stress, engineering strain does not satisfy certain coordinate transformation laws, which we will study in Chapter 9. Hence it is not a second-order tensor but is related to it as follows:

$$\text{tensor normal strains} = \text{engineering normal strains}; \qquad \text{tensor shear strains} = \frac{\text{engineering shear strains}}{2}$$

In Chapter 9 we shall see that the factor 1/2, which changes engineering shear strain to tensor shear strain, plays an important role in strain transformation.

2.6.1 Strain at a Point on a Line

In axial members we shall see that the displacement u is only a function of x. Hence the partial derivative in Equation (2.12a) becomes an ordinary derivative, and we obtain

$$\boxed{\varepsilon_{xx} = \frac{du}{dx}(x)} \tag{2.13}$$

If the displacement is given as a function of x, then we can obtain the strain as a function of x by differentiating. If strain is given as a function of x, then by integrating we can obtain the deformation between two points —that is, the relative displacement of two points. If we know the displacement of one of the points, then we can find the displacement of the other point. Alternatively stated, the integration of Equation (2.13) generates a constant of integration. To determine it, we need to know the displacement at a point on the line.

EXAMPLE 2.10

Calculations using the finite-element method (see Section 4.8) show that the displacement in a quadratic axial element is given by

$$u(x) = 125.0(x^2 - 3x + 8)10^{-6} \text{ cm}, \qquad 0 \le x \le 2 \text{ cm}$$

Determine the normal strain ε_{xx} at $x = 1$ cm.

PLAN

We can find the strain by using Equation at any x and obtain the final result by substituting the value of $x = 1$.

SOLUTION

Differentiating the given displacement, we obtain the strain as shown in Equation (E1).

$$\varepsilon_{xx}(x = 1) = \left.\frac{du}{dx}\right|_{x=1} = 125.0(2x - 3)10^{-6}\Big|_{x=1} = -125(10^{-6}) \tag{E1}$$

ANS. $\varepsilon_{xx}(x = 1) = -125\ \mu$

EXAMPLE 2.11

Figure 2.25 shows a bar that has axial strain $\varepsilon_{xx} = K(L - x)$ due to its own weight. K is a constant for a given material. Find the total extension of the bar in terms of K and L.

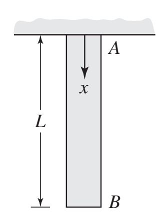


Figure 2.25 Bar in Example 2.11.

PLAN

The elongation of the bar corresponds to the displacement of point B. We start with Equation (2.13) and integrate to obtain the relative displacement of point B with respect to A. Knowing that the displacement at point A is zero, we obtain the displacement of point B.

SOLUTION

We substitute the given strain in Equation (2.13):

$$\varepsilon_{xx} = \frac{du}{dx} = K(L-x) \tag{E1}$$

Integrating Equation (E1) from point A to point B we obtain

$$\int_{u_A}^{u_B} du = \int_{x_A=0}^{x_B=L} K(L-x)\,dx \qquad \text{or} \qquad u_B - u_A = K\left(Lx - \frac{x^2}{2}\right)\Big|_0^L = K\left(L^2 - \frac{L^2}{2}\right) \tag{E2}$$

Since point A is fixed, the displacement $u_A = 0$ and we obtain the displacement of point B.

ANS. $u_B = (KL^2)/2$

COMMENTS

1. From strains we obtain deformation, that is relative displacement $u_B - u_A$. To get the absolute displacement we choose a point on the body that did not move.
2. We could integrate Equation (E1) to obtain $u(x) = K(Lx - x^2/2) + C_1$. Using the condition that the displacement u at $x = 0$ is zero, we obtain the integration constant $C_1 = 0$. We could then substitute $x = L$ to obtain the displacement of point B. The integration constant C_1 represents rigid-body translation, which we eliminate by fixing the bar to the wall.

Consolidate your knowledge

1. Explain in your own words deformation, strain, and their relationship without using equations.

QUICK TEST 1.1

Time: 15 minutes/Total: 20 points

Grade yourself using the answers in Appendix E. Each problem is worth 2 points.

1. What is the difference between displacement and deformation?
2. What is the difference between Lagrangian and Eulerian strains?
3. In decimal form, what is the value of normal strain that is equal to 0.3%?
4. In decimal form, what is the value of normal strain that is equal to 2000 μ?
5. Does the right angle increase or decrease with positive shear strains?
6. If the left end of a rod moves more than the right end in the negative x direction, will the normal strain be negative or positive? Justify your answer.
7. Can a 5% change in length be considered to be small normal strain? Justify your answer.
8. How many nonzero strain components are there in three dimensions?
9. How many nonzero strain components are there in plane strain?
10. How many independent strain components are there in plane strain?

PROBLEM SET 2.2

Strain components

2.67 A rectangle deforms into the colored shape shown in Figure P2.67. Determine ε_{xx}, ε_{yy}, and γ_{xy} at point A.

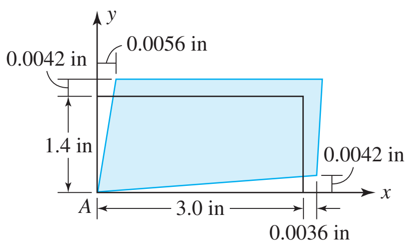


Figure P2.67

2.68 A rectangle deforms into the colored shape shown in Figure P2.68. Determine ε_{xx}, ε_{yy}, and γ_{xy} at point A.

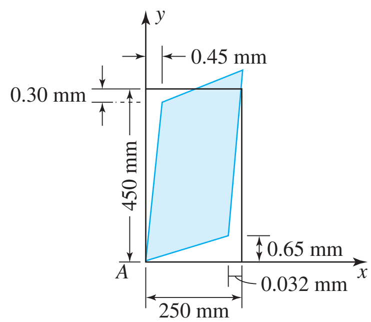


Figure P2.68

2.69 A rectangle deforms into the colored shape shown in Figure P2.69. Determine ε_{xx}, ε_{yy}, and γ_{xy} at point A.

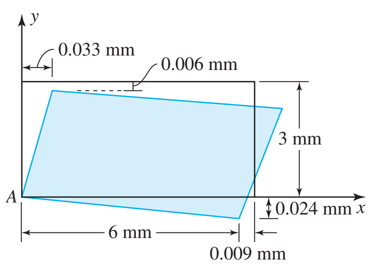


Figure P2.69

2.70 Displacements u and v in x and y directions, respectively, were measured by the Moiré interferometry method at many points on a body. The displacements of four points shown in Figure P2.70 are as give below. Determine the average values of the strain components ε_{xx}, ε_{yy}, and γ_{xy} at point A.

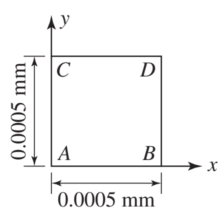


Figure P2.70

$$
\begin{aligned}
u_A &= 0.500 \mu\text{mm} & \quad \text{v}_A &= -1.000 \mu\text{mm} \\
u_B &= 1.125 \mu\text{mm} & \quad \text{v}_B &= -1.3125 \mu\text{mm} \\
u_C &= 0 & \quad \text{v}_C &= -1.5625 \mu\text{mm} \\
u_D &= 0.750 \mu\text{mm} & \quad \text{v}_D &= -2.125 \mu\text{mm}
\end{aligned}
$$

2.71 Displacements u and v in x and y directions, respectively, were measured by the Moiré interferometry method at many points on a body. The displacements of four points shown in Figure P2.70 are as given below. Determine the average values of the strain components ε_{xx}, ε_{yy}, and γ_{xy} at point A.

$$
\begin{aligned}
u_A &= 0.625 \mu\text{mm} & \quad \text{v}_A &= -0.3125 \mu\text{mm} \\
u_B &= 1.500 \mu\text{mm} & \quad \text{v}_B &= -0.5000 \mu\text{mm} \\
u_C &= 0.250 \mu\text{mm} & \quad \text{v}_C &= -1.125 \mu\text{mm} \\
u_D &= 1.250 \mu\text{mm} & \quad \text{v}_D &= -1.5625 \mu\text{mm}
\end{aligned}
$$

2.72 Displacements u and v in x and y directions, respectively, were measured by the Moiré interferometry method at many points on a body. The displacements of four points shown in Figure P2.70 are as given below. Determine the average values of the strain components ε_{xx}, ε_{yy}, and γ_{xy} at point A.

$$u_A = -0.500\mu\text{mm} \qquad v_A = -0.5625\mu\text{mm}$$
$$u_B = 0.250\mu\text{mm} \qquad v_B = -1.125\mu\text{mm}$$
$$u_C = -1.250\mu\text{mm} \qquad v_C = -1.250\mu\text{mm}$$
$$u_D = -0.375\mu\text{mm} \qquad v_D = -2.0625\mu\text{mm}$$

2.73 Displacements u and v in x and y directions, respectively, were measured by the Moiré interferometry method at many points on a body. The displacements of four points shown in Figure P2.70 are as given below. Determine the average values of the strain components ε_{xx}, ε_{yy}, and γ_{xy} at point A.

$$u_A = 0.250\mu\text{mm} \qquad v_A = -1.125\mu\text{mm}$$
$$u_B = 1.250\mu\text{mm} \qquad v_B = -1.5625\mu\text{mm}$$
$$u_C = -0.375\mu\text{mm} \qquad v_C = -2.0625\mu\text{mm}$$
$$u_D = 0.750\mu\text{mm} \qquad v_D = -2.7500\mu\text{mm}$$

Strain at a point

2.74 In a tapered circular bar that is hanging vertically, the axial displacement due to its weight was found to be

$$u(x) = \left(-19.44 + 1.44x - 0.01x^2 - \frac{933.12}{72 - x}\right)10^{-3} \text{ in.}$$

Determine the axial strain ε_{xx} at $x = 24$ in.

2.75 In a tapered rectangular bar that is hanging vertically, the axial displacement due to its weight was found to be

$$u(x) = [7.5(10^{-6})x^2 - 25(10^{-6})x - 0.15\ln(1 - 0.004x)] \text{ mm}$$

Determine the axial strain ε_{xx} at $x = 100$ mm.

2.76 The axial displacement in the quadratic one-dimensional finite element shown in Figure P2.76 is given below. Determine the strain at node 2.

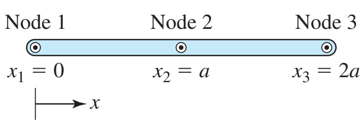


Figure P2.76

$$u(x) = \frac{u_1}{2a^2}(x-a)(x-2a) - \frac{u_2}{a^2}(x)(x-2a) + \frac{u_3}{2a^2}(x)(x-a)$$

2.77 The strain in the tapered bar due to the applied load in Figure P2.77 was found to be $\varepsilon_{xx} = 0.2/(40 - x)^2$. Determine the extension of the bar.

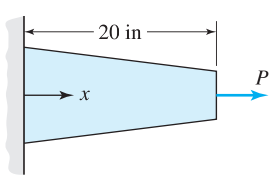


Figure P2.77

2.78 The axial strain in a bar of length L was found to be

$$\varepsilon_{xx} = \frac{KL}{(4L - 3x)} \qquad 0 \le x \le L$$

where K is a constant for a given material, loading, and cross-sectional dimension. Determine the total extension in terms of K and L.

2.79 The axial strain in a bar of length L due to its own weight was found to be

$$\varepsilon_{xx} = K\left[4L - 2x - \frac{8L^3}{(4L-2x)^2}\right] \qquad 0 \le x \le L,$$

where K is a constant for a given material and cross-sectional dimension. Determine the total extension in terms of K and L.

2.80 A bar has a tapered and a uniform section securely fastened, as shown in Figure P2.80. Determine the total extension of the bar if the axial strain in each section is

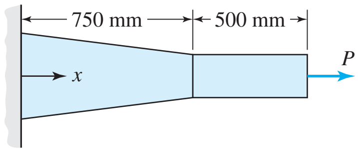


Figure P2.80

$$\varepsilon = \frac{1500 \times 10^3}{1875 - x}\,\mu, \qquad 0 \le x \le 750 \text{ mm}$$

$$\varepsilon = 1500\,\mu, \qquad 750 \text{ mm} \le x \le 1250 \text{ mm}$$

Stretch yourself

2.81 N axial bars are securely fastened together. Determine the total extension of the composite bar shown in Figure P2.81 if the strain in the i^{th} section is as given.

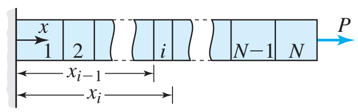


Figure P2.81

$$\varepsilon_i = a_i, \qquad x_{i-1} \le x \le x_i$$

2.82 True strain ε_T is calculated from $d\varepsilon_T = du/(L_0 + u)$, where u is the deformation at any given instant and L_0 is the original undeformed length. Thus the increment in true strain is the ratio of change in length at any instant to the length at that given instant. If ε represents engineering strain, show that at any instant the relationship between true strain and engineering strain is given by the following equation:

$$\varepsilon_T = \ln(1+\varepsilon) \tag{2.14}$$

2.83 The displacements in a body are given by

$$u = [0.5(x^2 - y^2) + 0.5xy](10^{-3}) \text{ mm} \qquad v = [0.25(x^2 - y^2) - xy](10^{-3}) \text{ mm}$$

Determine strains ε_{xx}, ε_{yy}, and γ_{xy} at $x = 5$ mm and $y = 7$ mm.

2.84 A metal strip is to be pulled and bent to conform to a rigid surface such that the length of the strip OA fits the arc OB of the surface shown in Figure P2.84. The equation of the surface is $f(x) = 0.04x^{3/2}$ in. and the length OA = 9 in. Determine the average normal strain in the metal strip.

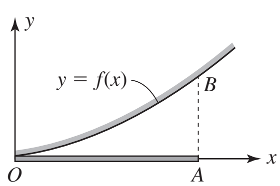


Figure P2.84

2.85 A metal strip is to be pulled and bent to conform to a rigid surface such that the length of the strip OA fits the arc OB of the surface shown in Figure P2.84. The equation of the surface is $f(x) = 625x^{3/2}$ μmm and the length OA = 200 mm. Determine the average normal strain in the metal strip.

Computer problems

2.86 A metal strip is to be pulled and bent to conform to a rigid surface such that the length of the strip OA fits the arc OB of the surface shown in Figure P2.84. The equation of the surface is $f(x) = (0.04x^{3/2} - 0.005x)$ in. and the length OA = 9 in. Determine the average normal strain in the metal strip. Use numerical integration.

2.87 Measurements made along the path of the stretch cord that is stretched over the canoe in Problem 2.4 (Figure P2.87) are shown in Table P2.87. The y coordinate was measured to the closest $\frac{1}{32}$ in. Between points A and B the cord path can be approximated by a straight line. Determine the average strain in the stretch cord if its original length it is 40 in. Use a spread sheet and approximate each 2-in. x interval by a straight line.

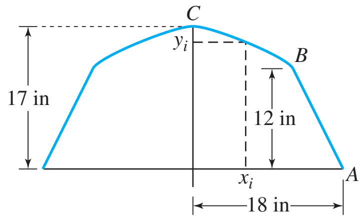


Figure P2.87

TABLE P2.87

x_i	y_i
0	17
2	$16\frac{30}{32}$
4	$16\frac{29}{32}$
6	$16\frac{19}{32}$
8	$16\frac{3}{32}$
10	$15\frac{16}{32}$
12	$14\frac{24}{32}$
14	$13\frac{28}{32}$
$x_B = 16$	$y_B = 12$
$x_A = 18$	$y_A = 0$

2.7* CONCEPT CONNECTOR

Like stress there are several definitions of strains. But unlike stress which evolved from intuitive understanding of strength to a mathematical definition, the development of concept of strain was mostly mathematical as described briefly in Section 2.7.1.

Displacements at different points on a solid body can be measured or analyzed by a variety of methods. One modern experimental technique is Moiré Fringe Method discussed briefly in Section 2.7.2.

2.7.1 History: The Concept of Strain

Normal strain, as a ratio of deformation over length, appears in experiments conducted as far back as the thirteenth century. Thomas Young (1773–1829) was the first to consider shear as an elastic strain, which he called *detrusion*. Augustin Cauchy (1789–1857), who introduced the concept of stress we use in this book (see Section 1.6.1), also introduced the mathematical definition of engineering strain given by Equations (2.12a) through (2.12f). The nonlinear Lagrangian strain written in tensor form was introduced by the English mathematician and physicist George Green (1793–1841) and is today called Green's strain tensor. The nonlinear Eulerian strain tensor, introduced in 1911 by E. Almansi, is also called Almansi's strain tensor. Green's and Almansi's strain tensors are often referred to as strain tensors in Lagrangian and Eulerian coordinates, respectively.

2.7.2 Moiré Fringe Method

Moiré fringe method is an experimental technique of measuring displacements that uses light interference produced by two equally spaced gratings. Figure 2.26 shows equally spaced parallel bars in two gratings. The spacing between the bars is called the *pitch*. Suppose initially the bars in the grating on the right overlap the spacings of the left. An observer on the right will be in a dark region, since no light ray can pass through both gratings. Now suppose that left grating moves, with a displacement less than the spacing between the bars. We will then have space between each pair of bars, resulting in regions of dark and light. These lines of light and dark lines are called *fringes*. When the left grating has moved through one pitch, the observer will once more be in the dark. By counting the number of times the regions of light and dark (i.e., the number of fringes passing this point) and multiplying by the pitch, we can obtain the displacement.

Note that any motion of the left grating parallel to the direction of the bars will not change light intensity. Hence displacements calculated from Moiré fringes are always perpendicular to the lines in the grating. We will need a grid of perpendicular lines to find the two components of displacements in a two-dimensional problem.

The left grating may be cemented, etched, printed, photographed, stamped, or scribed onto a specimen. Clearly, the order of displacement that can be measured depends on the number of lines in each grating. The right grating is referred to as the *reference grating*.

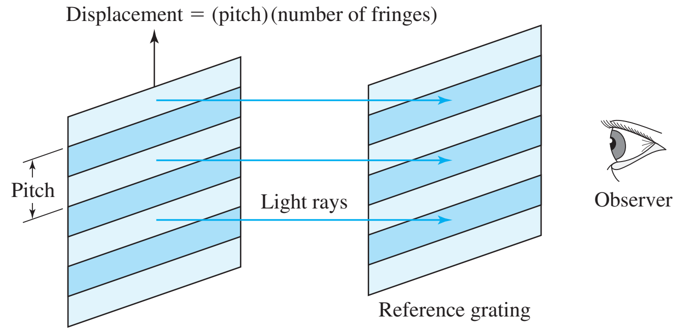


Figure 2.26 Destructive light interference by two equally spaced gratings.

Figure 2.26 illustrates light interference produced mechanical and is called *geometric Moiré method.* This method is used for displacement measurements in the range of 1 mm to as small as 10 μm, which corresponds to a grid of 1 to 100 lines per millimeter. In U.S. customary units the range is from 0.1 in. to as small as 0.001 in., corresponding to grids having from 10 to 1000 lines per inch.

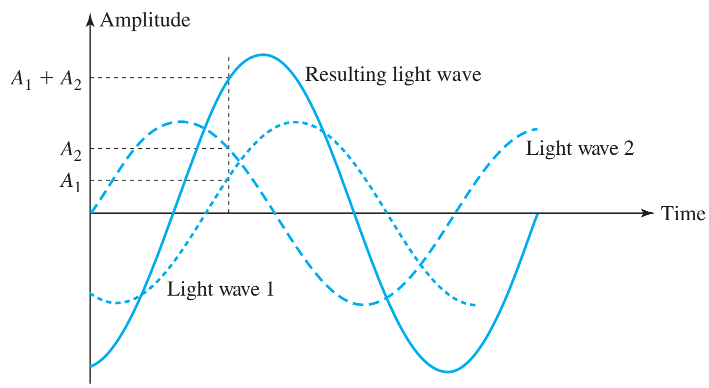
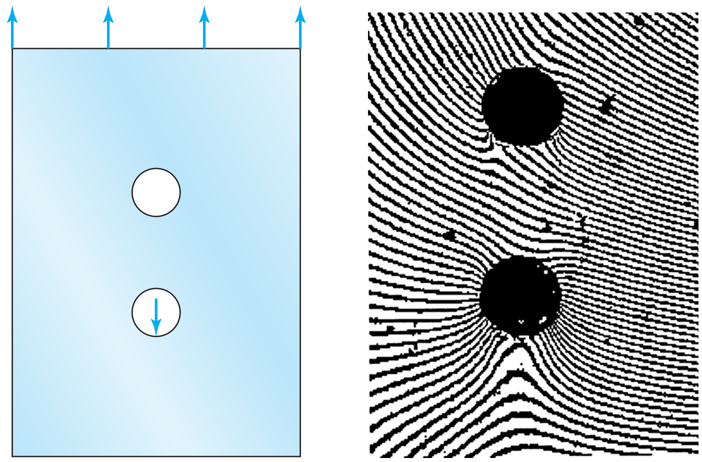


Figure 2.27 Superposition of two light waves.

Light interference can also be produced optically and techniques based on optical light interference are termed *optical interferometry.* Consider two light rays of the same frequency arriving at a point, as shown in Figure 2.27. The amplitude of the resulting light wave is the sum of the two amplitudes. If the crest of one light wave falls on the trough of another light wave, then the resulting amplitude will be zero, and we will have darkness at that point. If the crests of two waves arrive at the same time, then we will have light brighter than either of the two waves alone. This addition and subtraction, called *constructive* and *destructive* interference, is used in interferometry for measurements in a variety of ways.

In Moiré interferometry, for example, a reference grid may be created by the reflection of light from a grid fixed to the specimen, using two identical light sources. As the grid on the specimen moves, the reflective light and the incident light interfere constructively and destructively to produce Moiré fringes. Displacements as small as 10^{-5} in., corresponding to 100,000 lines per inch, can be measured. In the metric system, the order of displacements is 25×10^{-5} mm, which corresponds to 4000 lines per millimeter.

In an experiment to study mechanically fastened composites, load was applied on one end of the joint and equilibrated by applying a load on the lower hole, as shown in Figure 2.28*a*. Moiré fringes parallel to the applied load on the top plate are shown in Figure 2.28*b*.



(a) (b)

Figure 2.28 Deformation of a grid obtained from optical Moiré interferometry.

2.8 CHAPTER CONNECTOR

In this chapter we saw that the relation between displacement and strains is derived by studying the geometry of the deformed body. However, whenever we approximate a deformed body, we make assumptions regarding the displacements of points on the body. The simplest approach is to assume that each component of displacement is either a constant in the direction of coordinate axis, or else a linear function of the coordinate. From the displacements we can then obtain the strains.

The strain–displacement relation is independent of material properties. In the next chapter we shall introduce material properties and the relationship between stresses and strains. Thus, from displacements we first deduce the strains. From these we will deduce stress variations, from which we can find the internal forces. Finally, we relate the internal forces to external forces, as we did in Chapter 1. We shall see the complete logic in Chapter 3.

We will study strains again in Chapter 9, on strain transformation which relates strains in different coordinate systems. This is important as both experimental measurements and strains analyses are usually performed in a coordinate system chosen to simplify calculations. Developing a discipline of drawing deformed shapes has the same importance as drawing a free-body diagrams for calculating forces. These drawing provide an intuitive understanding of deformation and strain, as well as reduce mistakes in calculations.

POINTS AND FORMULAS TO REMEMBER

- The total movement of a point with respect to a fixed reference coordinate is called displacement.
- The relative movement of a point with respect to another point on the body is called deformation.
- The displacement of a point is the sum of rigid body motion and motion due to deformation.
- Lagrangian strain is computed from deformation by using the original undeformed geometry as the reference geometry.
- Eulerian strain is computed from deformation by using the final deformed geometry as the reference geometry.
- $\varepsilon = \dfrac{L_f - L_0}{L_0}$ (2.1) $\quad \varepsilon = \dfrac{\delta}{L_0}$ (2.2) $\quad \varepsilon = \dfrac{u_B - u_A}{x_B - x_A}$ (2.3)
- where ε is the average normal strain, L_0 is the original length of a line, L_f is the final length of a line, δ is the deformation of the line, and u_A and u_B are displacements of points x_A and x_B, respectively.
- Elongations result in positive normal strains. Contractions result in negative normal strains.
- $\gamma = \pi/2 - \alpha$ (2.4)

  where α is the final angle measured in radians and $\pi/2$ is the original right angle.

- Decreases in angle result in positive shear strain. Increases in angle result in negative shear strain.
- Small-strain approximation may be used for strains less than 0.01.
- Small-strain calculations result in linear deformation analysis.
- Small normal strains are calculated by using the deformation component in the original direction of the line element, regardless of the orientation of the deformed line element.
- In small shear strain (γ) calculations the following approximation may be used for the trigonometric functions:
- $\tan\gamma \approx \gamma \qquad \sin\gamma \approx \gamma \qquad \cos\gamma \approx 1$
- In small strain,
- $\delta = \bar{\mathbf{D}}_{AP} \cdot \bar{\mathbf{i}}_{AP}$ (2.7)
- where $\bar{\mathbf{D}}_{AP}$ is the deformation vector of the bar *AP* and $\bar{\mathbf{i}}_{AP}$ is the unit vector in the original direction of the bar *AP*.
- The same reference point must be used in the calculations of the deformation vector and the unit vector.

Average strain

$$\varepsilon_{xx} = \frac{\Delta u}{\Delta x} \qquad \gamma_{xy} = \gamma_{yx} = \frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x}$$
$$\varepsilon_{yy} = \frac{\Delta v}{\Delta y} \qquad \gamma_{yz} = \gamma_{zy} = \frac{\Delta v}{\Delta z} + \frac{\Delta w}{\Delta y}$$
$$\varepsilon_{zz} = \frac{\Delta w}{\Delta z} \qquad \gamma_{zx} = \gamma_{xz} = \frac{\Delta w}{\Delta x} + \frac{\Delta u}{\Delta z}$$

(2.9a) through (2.9i)

Strain at a point

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$
$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

(2.12a) through (2.12f)

- where u, v, and w are the displacements of a point in the x, y, and z directions, respectively.
- Shear strain is symmetric.
- In three dimensions there are nine strain components but only six are independent.
- In two dimensions there are four strain components but only three are independent.
- If u is only a function of x,

$$\varepsilon_{xx} = \frac{du(x)}{dx} \quad (2.13)$$