CHAPTER FIVE
TORSION OF SHAFTS

Learning objectives

1. Understand the theory, its limitations, and its applications in design and analysis of torsion of circular shafts.
2. Visualize the direction of torsional shear stress and the surface on which it acts.

When you ride a bicycle, you transfer power from your legs to the pedals, and through shaft and chain to the rear wheel. In a car, power is transferred from the engine to the wheel requiring many shafts that form the drive train such as shown in Figure 5.1*a*. A shaft also transfers torque to the rotor blades of a helicopter, as shown in Figure 5.1*b*. Lawn mowers, blenders, circular saws, drills—in fact, just about any equipment in which there is circular motion has shafts.

Any structural member that transmits torque from one plane to another is called a **shaft**. This chapter develops the simplest theory for torsion in circular shafts, following the logic shown in Figure 3.15, but subject to the limitations described in Section 3.13. We then apply the formulas to the design and analysis of statically determinate and indeterminate shafts.

(a) (b)

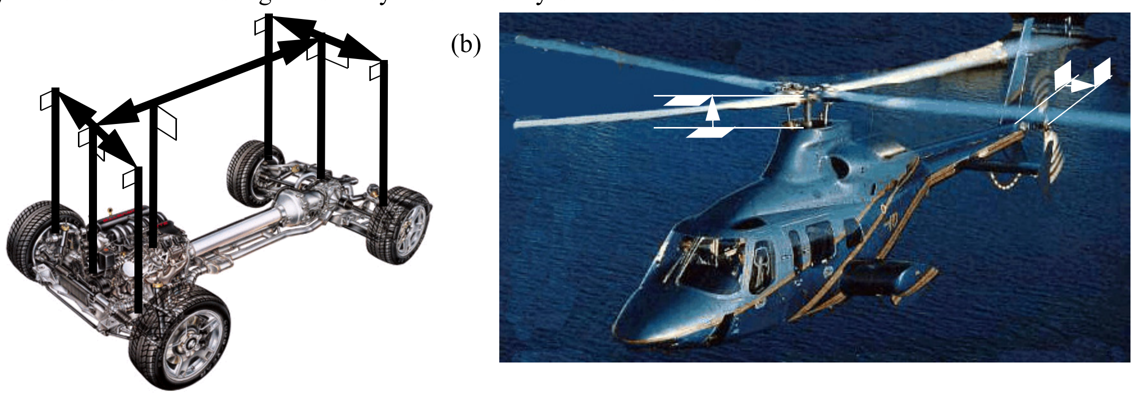


Figure 5.1 Transfer of torques between planes.

5.1 PRELUDE TO THEORY

As a prelude to theory, we consider several numerical examples solved using the logic discussed in Section 3.2. Their solution will highlight conclusions and observations that will be formalized in the development of the theory in Section 5.2.

- Example 5.1 shows the kinematics of shear strain in torsion. We apply the logic described in Figure 3.15, for the case of discrete bars attached to a rigid plate.
- Examples 5.2 and 5.3 extend the of calculation of shear strain to continuous circular shafts.
- Example 5.4 shows how the choice of a material model affects the calculation of internal torque. As we shall see the choice affects only the stress distribution, leaving all other equations unchanged. Thus the strain distribution, which is a kinematic relationship, is unaffected. So is static equivalency between shear stress and internal torque, and so are the equilibrium equations relating internal torques to external torques. Though we shall develop the simplest theory using Hooke's law, most of the equations here apply to more complex models as well.

EXAMPLE 5.1

The two thin bars of hard rubber shown in Figure 5.2 have shear modulus $G = 280$ MPa and cross-sectional area of 20 mm^2. The bars are attached to a rigid disc of 20-mm radius. The rigid disc is observed to rotate about its axis by an angle of 0.04 rad due to the applied torque T_{ext}. Determine the applied torque T_{ext}.

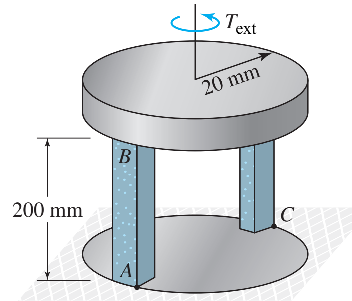


Figure 5.2 Geometry in Example 5.1.

PLAN

We can relate the rotation ($\Delta\phi = 0.04$) of the disc, the radius ($r = 0.02$ m) of the disc, and the length (0.2 m) of the bars to the shear strain in the bars as we did in Example 2.7. Using Hooke's law, we can find the shear stress in each bar. By assuming uniform shear stress in each bar, we can find the shear force. By drawing the free-body diagram of the rigid disc, we can find the applied torque T_{ext}.

SOLUTION

1. *Strain calculations:* Figure 5.3 shows an approximate deformed shape of the two bars. By symmetry the shear strain in bar C will be same as that in bar A. The shear strain in the bars can be calculated as in Example 2.7:

$$BB_1 = (0.02\text{ m})\,\Delta\phi = 0.0008\text{ m} \qquad \tan\gamma_A \approx \gamma_A = \frac{BB_1}{AB} = 0.004\text{ rad} \tag{E1}$$

$$\gamma_C = \gamma_A = 0.004\text{ rad} \tag{E2}$$

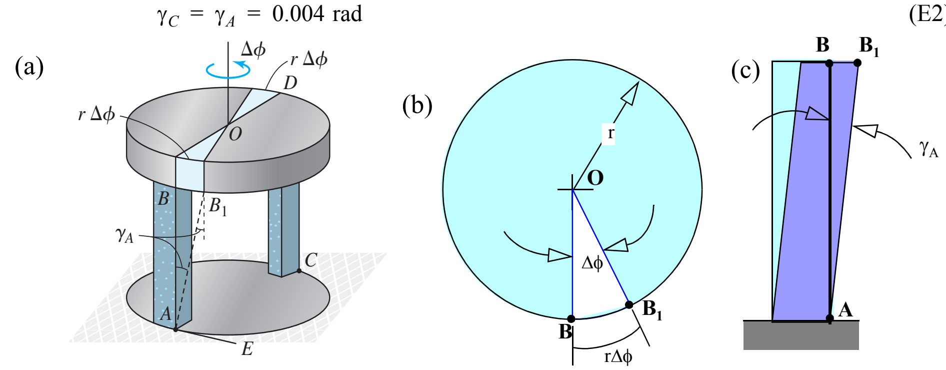


Figure 5.3 Exaggerated deformed geometry: (a) 3-D; (b) Top view; (c) Side view.

2. *Stress calculations:* From Hooke's law we can find the shear stresses as

$$\tau_A = G_A\gamma_A = [280(10^6)\text{ N/m}^2](0.004) = 1.12(10^6)\text{ N/m}^2 \tag{E3}$$

$$\tau_C = G_C\gamma_C = [280(10^6)\text{ N/m}^2](0.004) = 1.12(10^6)\text{ N/m}^2 \tag{E4}$$

3. *Internal forces:* We obtain the shear forces by multiplying the shear stresses by the cross-sectional area $A = 20 \times 10^{-6}\text{ m}^2$:

$$V_A = A_A\tau_A = [1.12(10^6)\text{ N/m}^2][20(10^{-6})\text{ m}^2] = 22.4\text{ N} \tag{E5}$$

$$V_C = A_C\tau_C = [1.12(10^6)\text{ N/m}^2][20(10^{-6})\text{ m}^2] = 22.4\text{ N} \tag{E6}$$

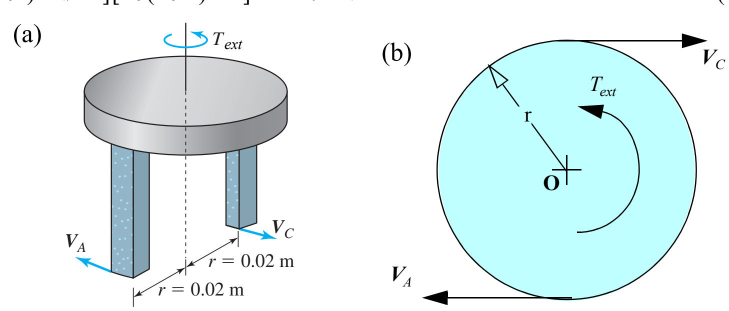


Figure 5.4 Free-body diagram: (a) 3-D; (b) Top view.

4. *External torque:* We draw the free-body diagram by making imaginary cuts through the bars, as shown in Figure 5.4. By equilibrium of moment about the axis of the disc through O, we obtain Equation (E7).

$$T_{ext} = rV_A + rV_C = (0.02 \text{ m})(22.4 \text{ N}) + (0.02 \text{ m})(22.4 \text{ N}) \tag{E7}$$

ANS. $T_{ext} = 0.896 \text{ N} \cdot \text{m}$

COMMENTS

1. In Figure 5.3 we approximated the arc BB_1 by a straight line, and we approximated the tangent function by its argument in Equation (E1). These approximations are valid only for small deformations and small strains. The net consequence of these approximations is that the shear strain along length AB_1 is uniform, as can be seen by the angle between any vertical line and line AB_1 at any point along the line.
2. The shear stress is assumed uniform across the cross section because of thin bars, but it is also uniform along the length because of the approximations described in comment 1.
3. The shear stress acts on a surface with outward normal in the direction of the length of the bar, which is also the axis of the disc. The shear force acts in the tangent direction to the circle of radius r. If we label the direction of the axis x, and the tangent direction θ, then the shear stress is represented by $\tau_{x\theta}$, as in Section 1.2
4. The sum in Equation (E7) can be rewritten as $\sum_{i=1}^{2} r\tau \; \Delta A_i$, where τ is the shear stress acting at the radius r, and ΔA_i is the cross-sectional area of the i^{th} bar. If we had n bars attached to the disc at the same radius, then the total torque would be given by $\sum_{i=1}^{n} r\tau \; \Delta A_i$. As we increase the number of bars n to infinity, the assembly approaches a continuos body. The cross-sectional area ΔA_i becomes the infinitesimal area dA, and the summation is replaced by an integral. We will formalize the observations in Section 5.1.1.
5. In this example we visualized a circular shaft as an assembly of bars. The next two examples further develop this idea.

EXAMPLE 5.2

A rigid disc of 20-mm diameter is attached to a circular shaft made of hard rubber, as shown in Figure 5.5. The left end of the shaft is fixed into a rigid wall. The rigid disc was rotated counterclockwise by 3.25°. Determine the average shear strain at point A.

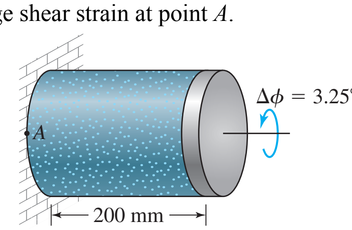


Figure 5.5 Geometry in Example 5.2.

PLAN

We can visualize the shaft as made up of infinitesimally thick bars of the type shown in Example 5.1. We relate the shear strain in the bar to the rotation of the disc, as we did in Example 5.1.

SOLUTION

We consider one line on the bar, as shown in Figure 5.6. Point B moves to point B_1. The right angle between AB and AC changes, and the change represents the shear strain γ. As in Example 5.1, we obtain the shear strain shown in Equation (E2):

$$\Delta\phi = \frac{3.25°\pi}{180°} = 0.05672 \text{ rad} \qquad BB_1 = r\,\Delta\phi = (10 \text{ mm})\,\Delta\phi = 0.5672 \text{ mm} \tag{E1}$$

$$\tan\gamma = \gamma = \frac{BB_1}{AB} = \frac{0.5672 \text{ mm}}{200 \text{ mm}} = 0.002836 \text{ rad} \tag{E2}$$

ANS. $\gamma = 2836 \; \mu\text{rad}$

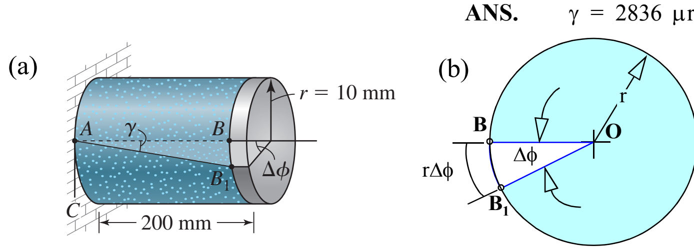


Figure 5.6 Deformed shape: (a) 3-D; (b) End view.

COMMENTS

1. As in Example 5.1, we assumed that the line AB remains straight. If the assumption were not valid, then the shear strain would vary in the axial direction.
2. The change of right angle that is being measured by the shear strain is the angle between a line in the axial direction and the tangent at any point. If we designate the axial direction x and the tangent direction θ (i.e., use polar coordinates), then the shear strain with subscripts will be $\gamma_{x\theta}$.

3. The value of the shear strain does not depend on the angular position as the problem is axisymmetric.
4. If we start with a rectangular grid overlaid on the shaft, as shown in Figure 5.7*a*, then each rectangle will deform by the same amount, as shown in Figure 5.7*b*. Based on the argument of axisymmetry, we will deduce this deformation for any circular shaft under torsion in the next section.

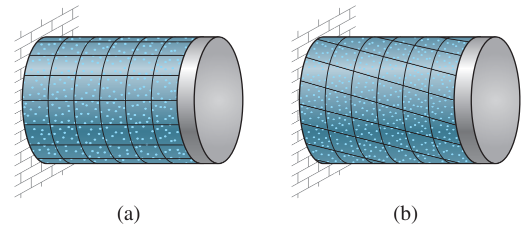


Figure 5.7 Deformation in torsion of (*a*) an un-deformed shaft. (*b*) a deformed shaft.

EXAMPLE 5.3

Three cylindrical shafts made from hard rubber are securely fastened to rigid discs, as shown in Figure 5.8. The radii of the shaft sections are $r_{AB} = 20$ mm, $r_{CD} = 15$ mm, and $r_{EF} = 10$ mm. If the rigid discs are twisted by the angles shown, determine the average shear strain in each section assuming the lines *AB*, *CD*, and *EF* remain straight.

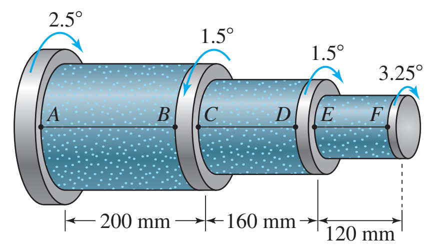


Figure 5.8 Shaft geometry in Example 5.3.

METHOD 1: PLAN

Each section of the shaft will undergo the deformation pattern shown in Figure 5.6, but now we need to account for the rotation of the disc at each end. We can analyze each section as we did in Example 5.2. In each section we can calculate the change of angle between the tangent and a line drawn in the axial direction at the point where we want to know the shear strain. We can then determine the sign of the shear strain using the definition of shear strain in Chapter 3.

SOLUTION

Label the left most disc as disc 1 and the rightmost disc, disc 4. The rotation of each disc in radians is as follows:

$$\phi_1 = \frac{2.5^\circ}{180^\circ}(3.142\ \text{rad}) = 0.0436\ \text{rad} \qquad \phi_2 = \frac{1.5^\circ}{180^\circ}(3.142\ \text{rad}) = 0.0262\ \text{rad}$$
$$\phi_3 = \frac{1.5^\circ}{180^\circ}(3.142\ \text{rad}) = 0.0262\ \text{rad} \qquad \phi_4 = \frac{3.25^\circ}{180^\circ}(3.142\ \text{rad}) = 0.0567\ \text{rad} \tag{E1}$$

Figure 5.9 shows approximate deformed shapes of the three segments,

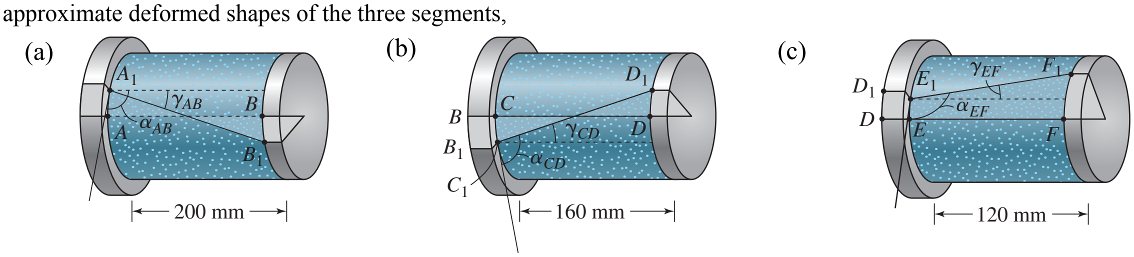


Figure 5.9 Approximate deformed shapes for Method 1 in Example 5.3 of segments (a) AB, (b) CD, and (c) EF.

Using Figure 5.9a we can find the shear strain in AB as

$$AA_1 = r_{AB}\phi_1 = (20\ \text{mm})(0.0436) = 0.872\ \text{mm} \qquad BB_1 = r_{AB}\phi_2 = (20\ \text{mm})(0.0262) = 0.524\ \text{mm} \tag{E2}$$

$$\tan|\gamma_{AB}| \approx |\gamma_{AB}| = \frac{AA_1 + BB_1}{AB} = \frac{0.872\ \text{mm} + 0.524\ \text{mm}}{200\ \text{mm}} \tag{E3}$$

The shear strain is positive as the angle γ_{AB} represents a decrease of angle from right angle.

ANS. $\gamma_{AB} = 6980\ \mu\text{rad}$

Using Figure 5.9b we can find the shear strain in CD as

$$CC_1 = r_{CD}\phi_2 = (15\text{ mm})(0.0262) = 0.393\text{ mm} \qquad DD_1 = r_{CD}\phi_3 = (15\text{ mm})(0.0262) = 0.393\text{ mm} \tag{E4}$$

$$\tan|\gamma_{CD}| \approx |\gamma_{CD}| = \frac{CC_1 + DD_1}{CD} = \frac{0.393\text{ mm} + 0.393\text{ mm}}{160\text{ mm}} \tag{E5}$$

The shear strain is negative as the angle γ_{CD} represents an increase of angle from right angle.

ANS. $\gamma_{CD} = -4913\ \mu\text{rad}$

Using Figure 5.9c we can find the shear strain in EF as

$$EE_1 = r_{EF}\phi_3 = (10\text{ mm})(0.0262) = 0.262\text{ mm} \qquad FF_1 = r_{EF}\phi_4 = (10\text{ mm})(0.0567) = 0.567\text{ mm} \tag{E6}$$

$$\tan|\gamma_{EF}| \approx |\gamma_{EF}| = \frac{FF_1 - EE_1}{EF} = \frac{0.567\text{ mm} - 0.262\text{ mm}}{120\text{ mm}} \tag{E7}$$

The shear strain is negative as the angle γ_{EF} represents an increase of angle from right angle.

ANS. $\gamma_{EF} = -2542\ \mu\text{rad}$

METHOD 2: PLAN

We assign a sign to the direction of rotation, calculate the relative deformation of the right disc with respect to the left disc, and analyze the entire shaft.

We draw an approximate deformed shape of the entire shaft, as shown in Figure 5.10. Let the counterclockwise rotation with respect to the x axis be positive and write each angle with the correct sign,

$$\phi_1 = -0.0436\text{ rad} \qquad \phi_2 = 0.0262\text{ rad} \qquad \phi_3 = -0.0262\text{ rad} \qquad \phi_4 = -0.0567\text{ rad} \tag{E8}$$

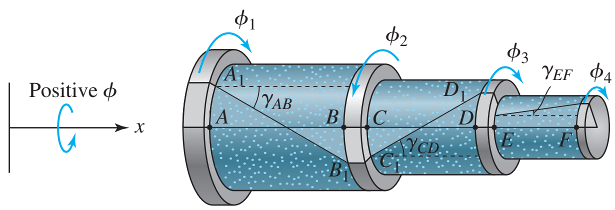


Figure 5.10 Shear strain calculation by Method 2 in Example 5.3.

We compute the relative rotation in each section and multiply the result by the corresponding section radius to obtain the relative movement of two points in a section. We then divide by the length of the section as we did in Example 5.2.

$$\Delta\phi_{AB} = \phi_2 - \phi_1 = 0.0698 \qquad \gamma_{AB} = \frac{r_{AB}\,\Delta\phi_{AB}}{AB} = \frac{(20\text{ mm})\ (0.0698)}{(200\text{ mm})} = 0.00698\text{ rad} \tag{E9}$$

$$\Delta\phi_{CD} = \phi_3 - \phi_2 = -0.0524 \qquad \gamma_{CD} = \frac{r_{CD}\,\Delta\phi_{CD}}{CD} = \frac{(15\text{ mm})\ (-0.0524)}{160\text{ mm}} = -0.004913\text{ rad} \tag{E10}$$

$$\Delta\phi_{EF} = \phi_4 - \phi_3 = -0.0305 \qquad \gamma_{EF} = \frac{r_{EF}\,\Delta\phi_{EF}}{EF} = \frac{(10\text{ mm})\ (-0.0305)}{120\text{ mm}} = -0.002542\text{ rad} \tag{E11}$$

ANS. $\gamma_{AB} = 6980\ \mu\text{rad} \qquad \gamma_{CD} = -4913\ \mu\text{rad} \qquad \gamma_{EF} = -2542\ \mu\text{rad}$

COMMENTS

1. Method 1 is easier to visualize, but the repetitive calculations can be tedious. Method 2 is more mathematical and procedural, but the repetitive calculations are easier. By solving the problems by method 2 but spending time visualizing the deformation as in method 1, we can reap the benefits of both.
2. We note that the shear strain in each section is directly proportional to the radius and the relative rotation of the shaft and inversely proportional to its length.

5.1.1 Internal Torque

Example 5.1 showed that the shear stress $\tau_{x\theta}$ can be replaced by an equivalent torque using an integral over the cross-sectional area. In this section we formalize that observation.

Figure 5.11 shows the shear stress distribution $\tau_{x\theta}$ that is to be replaced by an equivalent internal torque ***T***. Let ρ represent the radial coordinate, that is, the radius of the circle at which the shear stress acts. The moment at the center due to the shear stress on the differential area is $\rho\tau_{x\theta}\,dA$. By integrating over the entire area we obtain the total internal torque at the cross section.

$$\boxed{T = \int_A \rho \, dV = \int_A \rho \tau_{x\theta} \, dA} \quad (5.1)$$

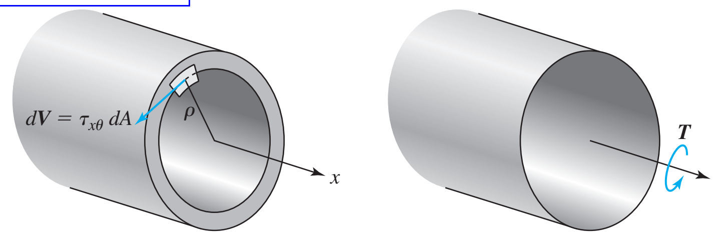


Figure 5.11 Statically equivalent internal torque.

Equation (5.1) is independent of the material model as it represents static equivalency between the shear stress on the entire cross section and the internal torque. If we were to consider a composite shaft cross section or nonlinear material behavior, then it would affect the value and distribution of $\tau_x\theta$ across the cross section. But Equation (5.1), relating $\tau_x\theta$ and $\boldsymbol{T}$, would remain unchanged. Examples 5.4 will clarify the discussion in this paragraph.

EXAMPLE 5.4

A homogeneous cross section made of brass and a composite cross section of brass and steel are shown in Figure 5.12. The shear moduli of elasticity for brass and steel are $G_B = 40$ GPa and $G_S = 80$ GPa, respectively. The shear strain in polar coordinates at the cross section was found to be $\gamma_{x\theta} = 0.08\rho$, where ρ is in meters. (a)Write expressions for $\tau_{x\theta}$ as a function of ρ and plot the shear strain and shear stress distributions across both cross sections. (b) For each of the cross sections determine the statically equivalent internal torques.

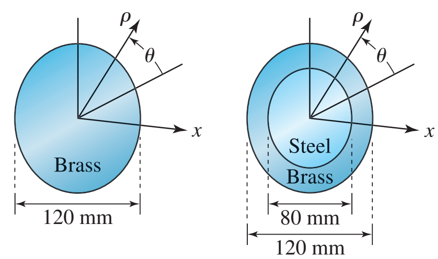


Figure 5.12 Homogeneous and composite cross sections in Example 5.4.

PLAN

(a) Using Hooke's law we can find the shear stress distribution as a function of ρ in each material. (b) Each of the shear stress distributions can be substituted into Equation (5.1) and the equivalent internal torque obtained by integration.

SOLUTION

(a) From Hooke's law we can write the stresses as

$$(\tau_{x\theta})_{\text{brass}} = [40(10^9)\ \text{N/m}^2](0.08\rho) = 3200\rho\ \text{MPa} \quad \text{(E1)}$$

$$(\tau_{x\theta})_{\text{steel}} = [80(10^9)\ \text{N/m}^2](0.08\rho) = 6400\rho\ \text{MPa} \quad \text{(E2)}$$

For the homogeneous cross section the stress distribution is as given in Equation (E1), but for the composite section it switches between Equation (E2) and Equation (E1), depending on the value of ρ. We can write the shear stress distribution for both cross sections as a function of ρ, as shown below.

Homogeneous cross section:

$$\tau_{x\theta} = 3200\rho\ \text{MPa} \qquad 0.00 \le \rho < 0.06 \quad \text{(E3)}$$

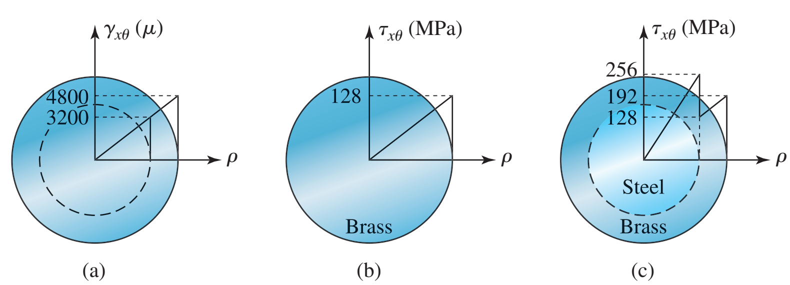


Figure 5.13 Shear strain and shear stress distributions in Example 5.4: (*a*) shear strain distribution; (*b*) shear stress distribution in homogeneous cross section; (c) shear stress distribution in composite cross section.

Composite cross section:

$$\tau_{x\theta} = \begin{cases} 6400\rho \text{ MPa} & 0.00 \le \rho < 0.04 \text{ m} \\ 3200\rho \text{ MPa} & 0.04 \text{ m} < \rho \le 0.06 \text{ m} \end{cases} \tag{E4}$$

The shear strain and the shear stress can now be plotted as a function of ρ, as shown in Figure 5.13(b). The differential area dA is the area of a ring of radius ρ and thickness $d\rho$, that is, $dA = 2\pi\rho\, d\rho$. Equation (5.1) can be written as

$$\boldsymbol{T} = \int_0^{0.06} \rho\tau_{x\theta}\,(2\pi\rho\, d\rho) \tag{E5}$$

Homogeneous cross section: Substituting Equation (E3) into Equation (E5) and integrating, we obtain the equivalent internal torque.

$$\boldsymbol{T} = \int_0^{0.06} \rho[3200\rho(10^6)\,](2\pi\rho\, d\rho) = [6400\pi(10^6)]\left(\frac{\rho^4}{4}\right)\Bigg|_0^{0.06} = 65.1(10^3)\text{ N}\cdot\text{m} \tag{E6}$$

ANS. $\boldsymbol{T} = 65.1$ kN·m

Composite cross section: Writing the integral in Equation (E5) as a sum of two integrals and substituting Equation (E3) we obtain the equivalent internal torque.

$$\boldsymbol{T} = \int_0^{0.06} \rho\tau_{x\theta}\,(2\pi\rho\, d\rho) = \underbrace{\int_0^{0.04} \rho\tau_{x\theta}\,(2\pi\rho\, d\rho)}_{\boldsymbol{T}_{\text{steel}}} + \underbrace{\int_{0.04}^{0.06} \rho\tau_{x\theta}\,(2\pi\rho\, d\rho)}_{\boldsymbol{T}_{\text{brass}}} \tag{E7}$$

$$T_{\text{steel}} = \int_0^{0.04} \rho[6400\rho(10^6)](2\pi\rho\, d\rho) = (12800\pi)(10^6)\left(\frac{\rho^4}{4}\right)\Bigg|_0^{0.04} = 25.7(10^3)\text{ N}\cdot\text{m} = 25.7\text{ kN}\cdot\text{m} \tag{E8}$$

$$T_{\text{brass}} = \int_{0.04}^{0.06} \rho[3200\rho(10^6)](2\pi\rho\, d\rho) = (6400\pi)(10^6)\left(\frac{\rho^4}{4}\right)\Bigg|_{0.04}^{0.06} = 52.3(10^3)\text{ N}\cdot\text{m} = 52.3\text{ kN}\cdot\text{m} \tag{E9}$$

$$T = T_{\text{steel}} + T_{\text{brass}} = 25.7\text{ kN}\cdot\text{m} + 52.3\text{ kN}\cdot\text{m} \tag{E10}$$

ANS. $T = 78$ kN·m

COMMENTS

1. The example demonstrates that although the shear strain varies linearly across the cross section, the shear stress may not. In this example we considered material non homogeneity. In a similar manner we can consider other models, such as elastic–perfectly plastic, or material models that have nonlinear stress–strain curves.
2. The material models dictate the shear stress distribution across the cross section, but once the stress distribution is known, Equation (5.1) can be used to find the equivalent internal torque, emphasizing that Equation (5.1) does not depend on the material model.

PROBLEM SET 5.1

5.1 A pair of 48-in. long bars and a pair of 60-in. long bars are symmetrically attached to a rigid disc at a radius of 2 in. at one end and built into the wall at the other end, as shown in Figure P5.1. The shear strain at point A due to a twist of the rigid disc was found to be 3000 μrad. Determine the magnitude of shear strain at point D.

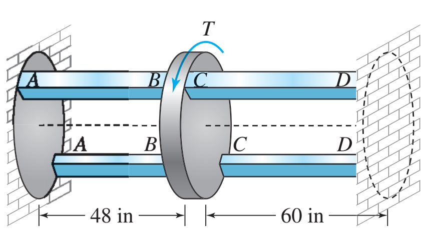


Figure P5.1

5.2 If the four bars in Problem 5.1 are made from a material that has a shear modulus of 12,000 ksi, determine the applied torque T on the rigid disc. The cross sectional areas of all bars are 0.25 in.^{2}.

5.3 If bars AB in Problem 5.1 are made of aluminum with a shear modulus $G_{al} = 4000$ ksi and bars CD are made of bronze with a shear modulus $G_{br} = 6500$ ksi, determine the applied torque T on the rigid disc. The cross-sectional areas of all bars are 0.25 in.^{2}.

5.4 Three pairs of bars are symmetrically attached to rigid discs at the radii shown in Figure P5.4. The discs were observed to rotate by angles $\phi_1 = 1.5°$, $\phi_2 = 3.0°$, and $\phi_3 = 2.5°$ in the direction of the applied torques T_1, T_2, and T_3, respectively. The shear modulus of the bars is 40 ksi and cross-sectional area is 0.04 in.^{2}. Determine the applied torques.

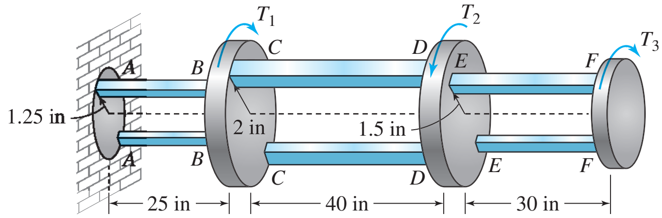


Figure P5.4

5.5 A circular shaft of radius r and length Δx has two rigid discs attached at each end, as shown in Figure P5.5. If the rigid discs are rotated as shown, determine the shear strain γ at point A in terms of r, Δx, and $\Delta\phi$, assuming that line AB remains straight, where $\Delta\phi = \phi_2 - \phi_1$.

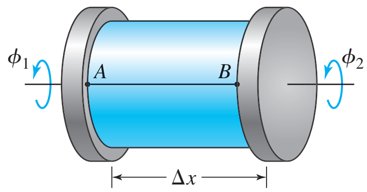


Figure P5.5

5.6 A hollow circular shaft made from hard rubber has an outer diameter of 4 in and an inner diameter of 1.5 in. The shaft is fixed to the wall on the left end and the rigid disc on the right hand is twisted, as shown in Figure P5.6. The shear strain at point A, which is on the outside surface, was found to be 4000 μrad. Determine the shear strain at point C, which is on the inside surface, and the angle of rotation. Assume that lines AB and CD remain straight during deformation.

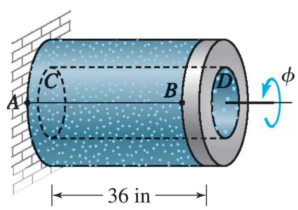


Figure P5.6

5.7 The magnitude of shear strains in the segments of the stepped shaft in Figure P5.7 was found to be γ_{AB} = 3000 μrad, γ_{CD} = 2500 μrad, and γ_{EF} = 6000 μrad. The radius of section AB is 150 mm, of section CD 70 mm, and of section EF 60 mm.Determine the angle by which each of the rigid discs was rotated.

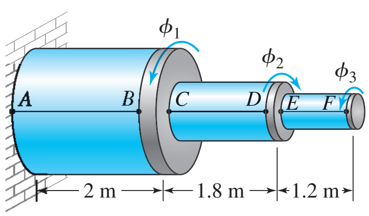


Figure P5.7

5.8 Figure P5.8 shows the cross section of a hollow aluminum (G= 26 GPa) shaft. The shear strain $\gamma_{x\theta}$ in polar coordinates at the section is $\gamma_{x\theta} = -0.06\rho$, where ρ is in meters. Determine the equivalent internal torque acting at the cross-section. Use d_i= 30 mm and d_o= 50 mm.

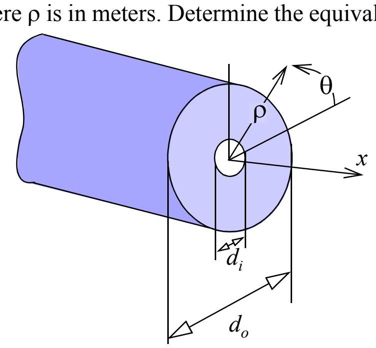


Figure P5.8

5.9 Figure P5.8 shows the cross section of a hollow aluminum ($G = 26$ GPa) shaft. The shear strain $\gamma_{x\theta}$ in polar coordinates at the section is $\gamma_{x\theta} = 0.05\rho$, where ρ is in meters. Determine the equivalent internal torque acting at the cross-section. Use $d_i = 40$ mm and $d_o = 120$ mm.

5.10 A hollow brass shaft ($G_B = 6500$ ksi) and a solid steel shaft ($G_S = 13{,}000$ ksi) are securely fastened to form a composite shaft, as shown in Figure P5.10.The shear strain in polar coordinates at the section is $\gamma_{x\theta} = 0.001\rho$, where ρ is in inches. Determine the equivalent internal torque acting at the cross section. Use $d_B = 4$ in. and $d_S = 2$ in.

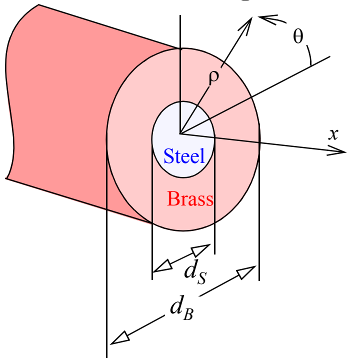


Figure P5.10

5.11 A hollow brass shaft ($G_B = 6500$ ksi) and a solid steel shaft ($G_S = 13{,}000$ ksi) are securely fastened to form a composite shaft, as shown in Figure P5.10.The shear strain in polar coordinates at the section is $\gamma_{x\theta} = -0.0005\rho$, where ρ is in inches. Determine the equivalent internal torque acting at the cross section. Use $d_B = 6$ in. and $d_S = 4$ in.

5.12 A hollow brass shaft ($G_B = 6500$ ksi) and a solid steel shaft ($G_S = 13{,}000$ ksi) are securely fastened to form a composite shaft, as shown in Figure P5.10.The shear strain in polar coordinates at the section is $\gamma_{x\theta} = 0.002\rho$, where ρ is in inches. Determine the equivalent internal torque acting at the cross section. Use $d_B = 3$ in. and $d_S = 1$ in.

5.13 A hollow titanium shaft ($G_{Ti} = 36$ *GPa*) and a hollow aluminum shaft ($G_{Al} = 26$ *GPa)* are securely fastened to form a composite shaft shown in Figure P5.13. The shear strain in polar coordinates at the section is $\gamma_{x\theta} = 0.04\rho$, where ρ is in meters. Determine the equivalent internal torque acting at the cross section. Use $d_i = 50$ mm, $d_{Al} = 90$ mm, and $d_{Ti} = 100$ mm.

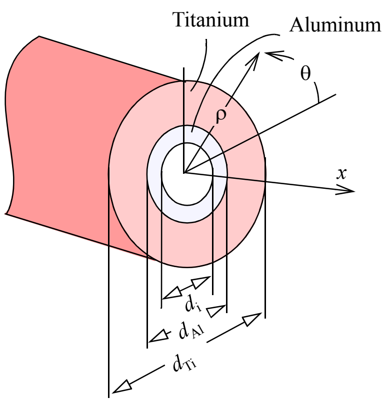


Figure P5.13

Stretch Yourself

5.14 A circular shaft made from elastic - perfectly plastic material has a torsional shear stress distribution across the cross section shown in Figure P5.14. Determine the equivalent internal torque.

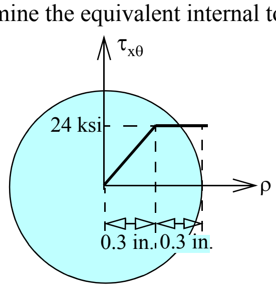


Figure P5.14

5.15 A solid circular shaft of 3-in. diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho\ (10^{-3})$, where ρ is the radial coordinate measured in inches. The shaft is made from an elastic–perfectly plastic material, which has a yield stress $\tau_{yield} = 18$ ksi and a shear modulus $G = 12{,}000$ ksi. Determine the equivalent internal torque. (See Problem 3.144).

5.16 A solid circular shaft of 3-in. diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho\ (10^{-3})$, where ρ is the radial coordinate measured in inches.The shaft is made form a bilinear material as shown in Figure 3.40. The material has a yield stress $\tau_{yield} = 18$ ksi and shear moduli $G_1 = 12{,}000$ ksi and $G_2 = 4800$ ksi. Determine the equivalent internal torque.(See Problem 3.145).

5.17 A solid circular shaft of 3-in. diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho\ (10^{-3})$, where ρ is the radial coordinate measured in inches.The shaft material has a stress–strain relationship given by $\tau = 243\gamma^{0.4}$ ksi. Determine the equivalent internal torque. (See Problem 3.146).

5.18 A solid circular shaft of 3-in diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho\ (10^{-3})$, where ρ is the radial coordinate measured in inches. The shaft material has a stress–strain relationship given by $\tau = 12{,}000\gamma - 120{,}000\gamma^2$ ksi. Determine the equivalent internal torque. (See Problem 3.147).

5.2 THEORY OF TORSION OF CIRCULAR SHAFTS

In this section we develop formulas for deformation and stress in a circular shaft. We will follow the procedure in Section 5.1 but now with variables in place of numbers. The theory will be developed subject to the following limitations:

1. The length of the member is significantly greater than the greatest dimension in the cross section.
2. We are away from the regions of stress concentration.
3. The variation of external torque or change in cross-sectional areas is gradual except in regions of stress concentration.
4. External torques are not functions of time; that is, we have a static problem. (See Problems 5.55 and 5.56 for dynamic problems.)
5. The cross section is circular. This permits us to use arguments of axisymmetry in deducing deformation.

Figure 5.14 shows a circular shaft that is loaded by external torques T_1 and T_2 at each end and an external distributed torque $t(x)$, which has units of torque per unit length. The radius of the shaft $R(x)$ varies as a function of x. We expect that the internal torque $\boldsymbol{T}$ will be a function of x. ϕ_1 and ϕ_2 are the angles of rotation of the imaginary cross sections at x_1 and x_2, respectively.

The objectives of the theory are:

1. To obtain a formula for the relative rotation $\phi_2 - \phi_1$ in terms of the internal torque $\boldsymbol{T}$.
2. To obtain a formula for the shear stress $\tau_{x\theta}$ in terms of the internal torque $\boldsymbol{T}$.

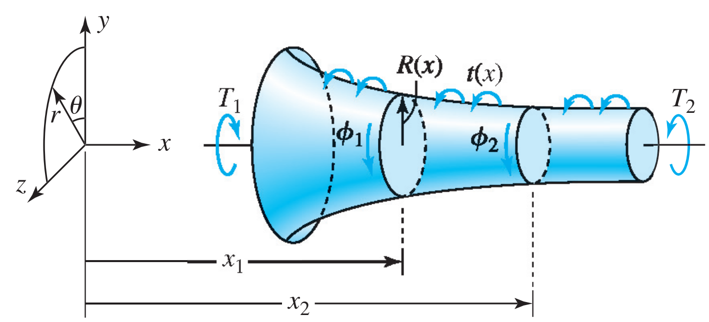


Figure 5.14 Circular shaft.

To account for the variations in $t(x)$ and $R(x)$ we will take $\Delta x = x_2 - x_1$ as an infinitesimal distance in which these quantities can be treated as constants. The deformation behavior across the cross section will be approximated. The logic shown in Figure 5.15 and discussed in Section 3.2 will be used to develop the simplest theory for the torsion of circular shafts members. Assumptions will be identified as we move from one step to the next. These assumptions are the points at which complexities can be added to the theory, as discussed in the examples and Stretch Yourself problems.

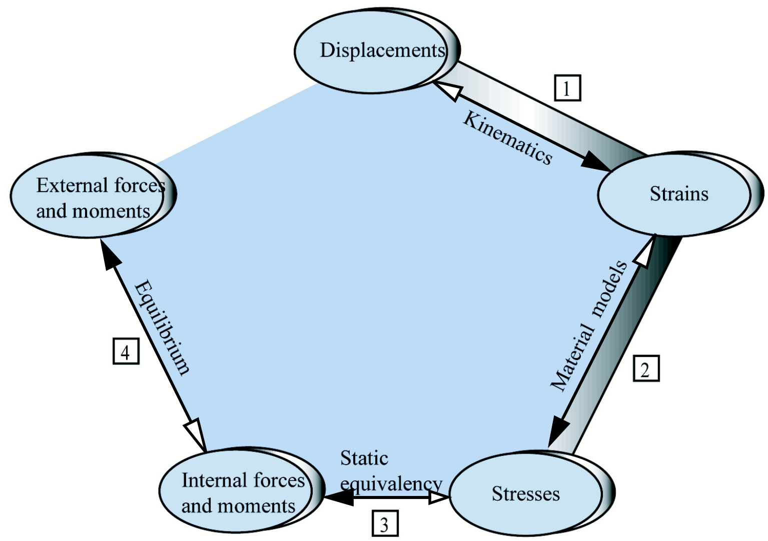


Figure 5.15 The logic of the mechanics of materials.

5.2.1 Kinematics

In Example 5.1 the shear strain in a bar was related to the rotation of the disc that was attached to it. In Example 5.2 we remarked that a shaft could be viewed as an assembly of bars. Three assumptions let us simulate the behavior of a cross section as a rotating rigid plate:

Assumption 1 Plane sections perpendicular to the axis remain plane during deformation.

Assumption 2 On a cross section, all radial lines rotate by equal angles during deformation.

Assumption 3 Radial lines remain straight during deformation.

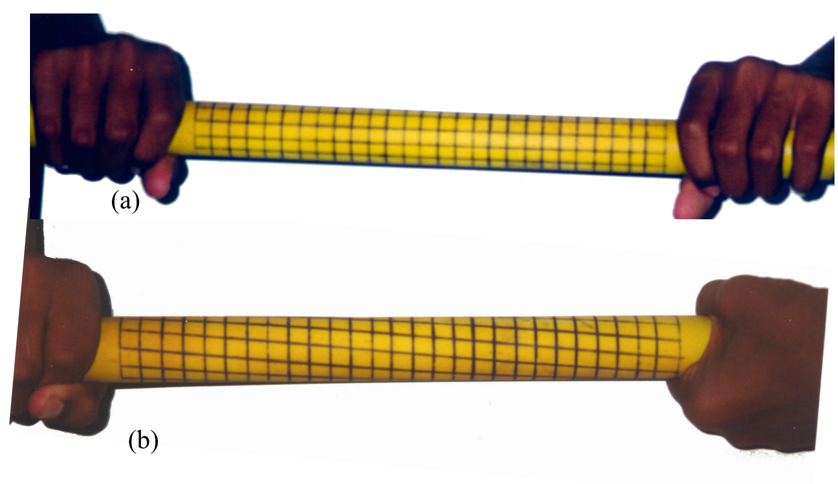


Figure 5.16 Torsional deformation: (*a*) original grid; (*b*) deformed grid. (Courtesy of Professor J. B. Ligon.)

Figure 5.16 shows a circular rubber shaft with a grid on the surface that is twisted by hand. The edges of the circles remain vertical lines during deformation. This observation confirms the validity of Assumption 1. Axial deformation due to torsional loads is called **warping**. Thus, circular shafts do not warp. Shafts with noncircular cross section warp, and this additional deformation leads to additional complexities. (See Problem 5.53).

The axisymmetry of the problem implies that deformation must be independent of the angular rotation. Thus, all radials lines must behave in exactly the same manner irrespective of their angular position, thus, Assumptions 2 and 3 are valid for circular shafts. Figure 5.17 shows that all radial lines rotate by the same **angle of twist** ϕ. We note that if all lines rotate by equal amounts on the cross section, then ϕ does not change across the cross section and hence can only be a function of x

$$\phi = \phi(x) \tag{5.2}$$

Sign Convention: ϕ is considered positive counterclockwise with respect to the x axis.

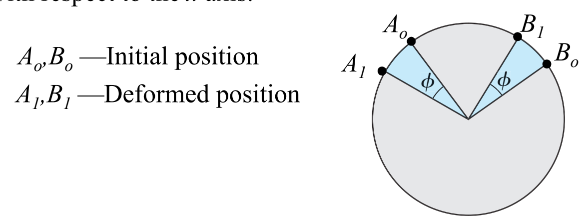


Figure 5.17 Equal rotation of all radial lines.

The shear strain of interest to us is the measure of the angle change between the axial direction and the tangent to the circle in Figure 5.16. If we use polar coordinates, then we are interested in the change in angle which is between the x and θ directions— in other words, $\gamma_{x\theta}$.

Assumptions 1 through 3 are analogous to viewing each cross section in the shaft as a rigid disc that rotates about its own axis. We can then calculate the shear strain as in Example 5.2, provided we have small deformation and strain.

Assumption 4 Strains are small.

We consider a shaft with radius ρ and length Δx in which the right section with respect to the left section is rotated by an angle $\Delta\phi$, as shown in Figure 5.18*a*. Using geometry we obtain the shear strain expression.

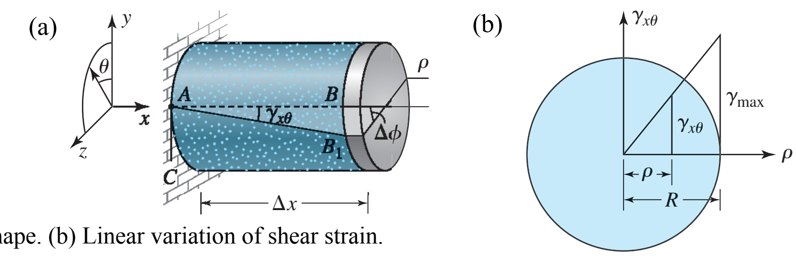


Figure 5.18 Shear strain in torsion. (a) Deformed shape. (b) Linear variation of shear strain.

$$\tan\gamma_{x\theta} \approx \gamma_{x\theta} = \lim_{AB \to 0}\left(\frac{BB_1}{AB}\right) = \lim_{\Delta x \to 0}\left(\frac{\rho\Delta\phi}{\Delta x}\right) \quad \text{or}$$

$$\boxed{\gamma_{x\theta} = \rho\frac{d\phi}{dx}} \tag{5.3}$$

where ρ is the radial coordinate of a point on the cross section. The subscripts x and θ emphasize that the change in angle is between the axial and tangent directions, as shown in Figure 5.18*a*. The quantity $d\phi/dx$ is called the **rate of twist**. It is a function of x only, because ϕ is a function of x only.

Equation (5.3) was derived from purely geometric considerations. If Assumptions 1 through 4 are valid, then Equation (5.3) is independent of the material. Equation (5.3) shows that the shear strain is a *linear function* of the radial coordinate ρ and reaches the maximum value γ_{max} at the outer surface ($\rho = \rho_{max} = R$), as shown in Figure 5.18*a*. Equation (5.4), an alternative form for shear strain, can be derived using similar triangles.

$$\gamma_{x\theta} = \frac{\gamma_{max}\rho}{R} \tag{5.4}$$

5.2.2 Material Model

Our motivation is to develop a simple theory for torsion of circular shafts. Thus we make assumptions regarding material behavior that will permit us to use the simplest material model given by Hooke's law.

Assumption 5 The material is linearly elastic.[1]

Assumption 6 The material is isotropic.

Substituting Equation (5.3) into Hooke's law, that is, $\tau = G\gamma$, we obtain

$$\tau_{x\theta} = G\rho\frac{d\phi}{dx} \tag{5.5}$$

Noting that θ is positive in the counterclockwise direction with respect to the x axis, we can represent the shear stress due to torsion on a stress element as shown in Figure 5.19. Also shown in Figure 5.19 are aluminum and wooden shafts that broke in torsion. The shear stress component that exceeds the shear strength in aluminum is $\tau_{x\theta}$. The shear strength of wood is weaker along the surface parallel to the grain, which for shafts is in the longitudinal direction. Thus $\tau_{\theta x}$ causes the failure in wooden shafts. The two failure surfaces highlight the importance of visualizing the torsional shear stress element.

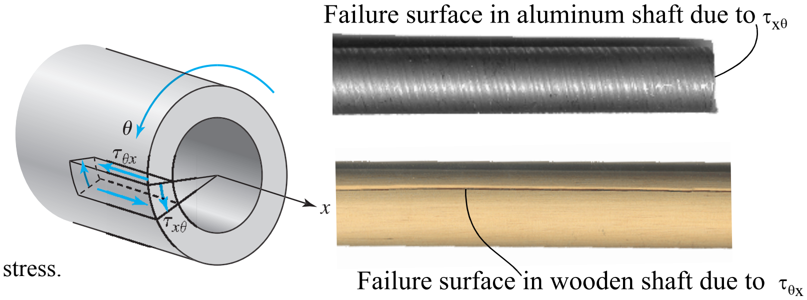


Figure 5.19 Stress element showing torsional shear stress.

5.2.3 Torsion Formulas

Substituting Equation (5.5) into Equation (5.1) and noting that $d\phi/dx$ is a function of x only, we obtain

$$\boldsymbol{T} = \int_A G\rho^2\frac{d\phi}{dx}\,dA = \frac{d\phi}{dx}\int_A G\rho^2\,dA \tag{5.6}$$

To simplify further, we would like to take G outside the integral, which implies that G cannot change across the cross section.

Assumption 7 The material is homogeneous across the cross section.[2]

From Equation (5.6) we obtain

$$\boldsymbol{T} = G\frac{d\phi}{dx}\int_A \rho^2\,dA = GJ\frac{d\phi}{dx} \tag{5.7}$$

where J is the **polar moment of inertia** for the cross section. As shown in Example 5.5, J for a circular cross section of radius R or diameter D is given by

$$\boxed{J = \int_A \rho^2\,dA = \frac{\pi}{2}R^4 = \frac{\pi}{32}D^4} \tag{5.8}$$

Equation (5.7) can be written as

$$\boxed{\frac{d\phi}{dx} = \frac{\boldsymbol{T}}{GJ}} \tag{5.9}$$

The higher the value of GJ, the smaller will be the deformation ϕ for a given value of the internal torque. Thus the rigidity of the shaft increases with the increase in GJ. A shaft may be made more rigid either by choosing a stiffer material (higher value of G) or by increasing the polar moment of inertia. The quantity GJ is called **torsional rigidity**.

Substituting Equation (5.9) into Equation (5.5), we obtain

[1]See Problems 5.50 through 5.52 for nonlinear material behavior.
[2]In Problem. 5.49 this assumption is not valid.

$$\boxed{\tau_{x\theta} = \frac{T\rho}{J}} \tag{5.10}$$

The quantities T and J do not vary across the cross section. Thus the shear stress varies *linearly* across the cross section with ρ as shown in Figure 5.20. For a solid shaft, it is zero at the center where $\rho = 0$ and reaches a *maximum value on the outer surface* of the shaft where $\rho = R$,.

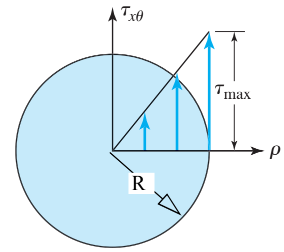


Figure 5.20 Linear variation of torsional shear stress.

Let the angle of rotation of the cross section at x_1 and x_2 be ϕ_1 and ϕ_2, respectively. By integrating Equation (5.9) we can obtain the relative rotation as:

$$\phi_2 - \phi_1 = \int_{\phi_1}^{\phi_2} d\phi = \int_{x_1}^{x_2} \frac{T}{GJ}\, dx \tag{5.11}$$

To obtain a simple formula we would like to take the three quantities T, G, and J outside the integral, which means that these quantities should not change with x. To achieve this simplicity we make the following assumptions:

Assumption 8 The material is homogeneous between x_1 and x_2. (G is constant)

Assumption 9 The shaft is not tapered between x_1 and x_2. (J is constant)

Assumption 10 The external (and hence also the internal) torque does not change with x between x_1 and x_2. (T is constant)

If Assumptions 8 through 10 are valid, then T, G, and J are constant between x_1 and x_2, and from Equation (5.11) we obtain

$$\boxed{\phi_2 - \phi_1 = \frac{T(x_2 - x_1)}{GJ}} \tag{5.12}$$

In Equation (5.12) points x_1 and x_2 must be chosen such that neither T, G, nor J change between these points.

5.2.4 Sign Convention for Internal Torque

The shear stress was replaced by a statically equivalent internal torque using Equation (5.1). The shear stress $\tau_{x\theta}$ is positive on two surfaces. Hence the equivalent internal torque is positive on two surfaces, as shown in Figure 5.21. When we make the imaginary cut to draw the free-body diagram, then the internal torque must be drawn in the positive direction if we want the formulas to give the correct signs.

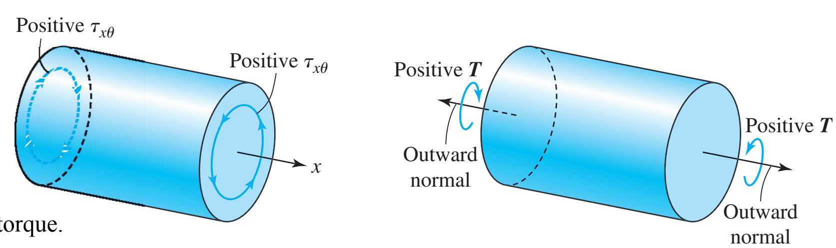


Figure 5.21 Sign convention for positive internal torque.

Sign Convention: Internal torque is considered positive counterclockwise with respect to the outward normal to the imaginary cut surface.

T may be found in either of two ways, as described next and elaborated further in Example 5.6.

1. T is always drawn counterclockwise with respect to the outward normal of the imaginary cut, as per our sign convention. The equilibrium equation is then used to get a positive or negative value for T. The sign for relative rotation obtained from Equation (5.12) is positive counterclockwise with respect to the x axis. The direction of shear stress can be determined using the subscripts, as in Section 1.3.

2. ***T*** is drawn at the imaginary cut to equilibrate the external torques. Since inspection is used to determine the direction of ***T***, the direction of relative rotation in Equation (5.12) and the direction of shear stress $\tau_{x\theta}$ in Equation (5.10) must also be determined by inspection.

5.2.5 Direction of Torsional Stresses by Inspection.

The significant shear stress in the torsion of circular shafts is $\tau_{x\theta}$. All other stress components can be neglected provided the ratio of the length of the shaft to its diameter is on the order of 10 or more.

Figure 5.22*a* shows a segment of a shaft under torsion containing point *A*. We visualize point *A* on the left segment and consider the stress element on the left segment. The left segment rotates clockwise in relation to the right segment. This implies that point *A*, which is part of the left segment, is moving upward on the shaded surface. Hence the shear stress, like friction, on the shaded surface will be downward. We know that a pair of symmetric shear stress components points toward or away from the corner. From the symmetry, the shear stresses on the rest of the surfaces can be drawn as shown.

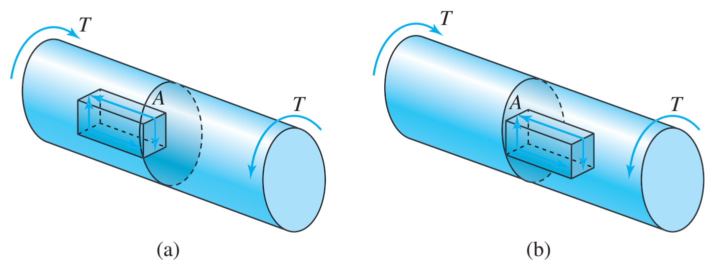


Figure 5.22 Direction of shear stress by inspection.

Suppose we had considered point *A* on the right segment of the shaft. In such a case we consider the stress element as part of the right segment, as shown in Figure 5.22*b*. The right segment rotates counterclockwise in relation to the left segment. This implies that point *A*, which is part of the right segment, is moving down on the shaded surface. Hence the shear stress, like friction, will be upward. Once more using the symmetry of shear stress components, the shear stress on the remaining surfaces can be drawn as shown.

In visualizing the stress surface, we need not draw the shaft segments in Figure 5.22. But care must be taken to identify the surface on which the shear stress is being considered. The shear stress on the adjoining imaginary surfaces have opposite direction. However, irrespective of the shaft segment on which we visualize the stress element, we obtain the same stress element, as shown in Figure 5.22. This is because the two stress elements shown represent the same point *A*.

An alternative way of visualizing torsional shear stress is to think of a coupling at an imaginary section and to visualize the shear stress directions on the bolt surfaces, as shown in Figure 5.23. Once the direction of the shear stress on the bolt surface is visualized, the remaining stress elements can be completed using the symmetry of shear stresses

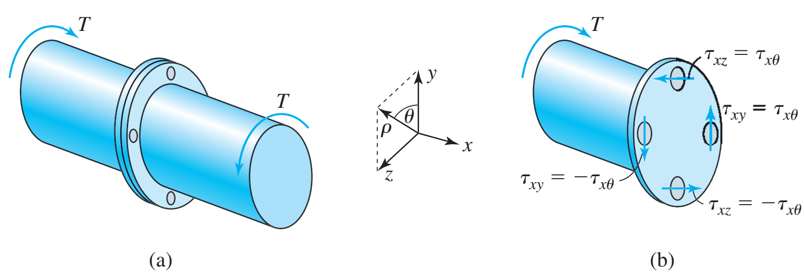


Figure 5.23 Torsional shear stresses.

After having obtained the torsional shear stress, either by using subscripts or by inspection, we can examine the shear stresses in Cartesian coordinates and obtain the stress components with correct signs, as shown in Figure 5.23*b*. This process of obtaining stress components in Cartesian coordinates will be important when we consider stress and strain transformation equations in Chapters 8 and 9, where we will relate stresses and strains in different coordinate systems.

The shear strain can be obtained by dividing the shear stress by *G*, the shear modulus of elasticity.

Consolidate your knowledge

1. Identify five examples of circular shafts from your daily life.
2. With the book closed, derive Equations 5.10 and 5.12, listing all the assumptions as you go along.

EXAMPLE 5.5

The two shafts shown in Figure 5.24 are of the same material and have the same amount of material cross-sectional areas A. Show that the hollow shaft has a larger polar moment of inertia than the solid shaft.

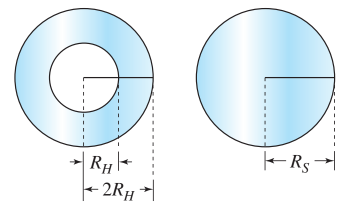


Figure 5.24 Hollow and solid shafts of Example 5.5.

PLAN

We can find the values of R_H and R_S in terms of the cross-sectional area A. We can then substitute these radii in the formulas for polar area moment to obtain the polar area moments in terms of A.

SOLUTION

We can calculate the radii R_H and R_S in terms of the cross sectional area A as

$$A_H = \pi[(2R_H)^2 - R_H^2] = A \quad \text{or} \quad R_H^2 = \frac{A}{3\pi} \quad \text{and} \quad A_S = \pi R_S^2 = A \quad \text{or} \quad R_S^2 = \frac{A}{\pi} \tag{E1}$$

The polar area moment of inertia for a hollow shaft with inside radius R_i and outside radius R_o can be obtained as

$$J = \int_A \rho^2\, dA = \int_{R_i}^{R_o} \rho^2(2\pi\rho)\, d\rho = \frac{\pi}{2}\rho^4\bigg|_{R_i}^{R_o} = \frac{\pi}{2}(R_o^4 - R_i^4) \tag{E2}$$

For the hollow shaft $R_o = 2R_H$ and $R_i = R_H$, whereas for the solid shaft $R_o = R_S$ and $R_i = 0$. Substituting these values into Equation (E2), we obtain the two polar area moments.

$$J_H = \frac{\pi}{2}[(2R_H)^4 - R_H^4] = \frac{15}{2}\pi R_H^4 = \frac{15}{2}\pi\left(\frac{A}{3\pi}\right)^2 = \frac{5}{6}\frac{A^2}{\pi} \quad \text{and} \quad J_S = \frac{\pi}{2}R_S^4 = \frac{\pi}{2}\left(\frac{A}{\pi}\right)^2 = \frac{A^2}{2\pi} \tag{E3}$$

Dividing J_H by J_S we obtain

$$\frac{J_H}{J_S} = \frac{5}{3} = 1.67 \tag{E4}$$

ANS. As $J_H > J_S$ the polar moment for the hollow shaft is greater than that of the solid shaft for the same amount of material.

COMMENT

1. The hollow shaft has a polar moment of inertia of 1.67 times that of the solid shaft for the same amount of material. Alternatively, a hollow shaft will require less material (lighter in weight) to obtain the same polar moment of inertia. This reduction in weight is the primary reason why metal shafts are made hollow. Wooden shafts, however, are usually solid as the machining cost does not justify the small saving in weight.

EXAMPLE 5.6

A solid circular steel shaft (G_s = 12,000 ksi) of variable diameter is acted upon by torques as shown in Figure 5.25. The diameter of the shaft between wheels A and B and wheels C and D is 2 in., and the diameter of the shaft between wheels B and C is 4 in. Determine: (a) the rotation of wheel D with respect to wheel A; (b) the magnitude of maximum torsional shear stress in the shaft; (c) the shear stress at point E. Show it on a stress cube.

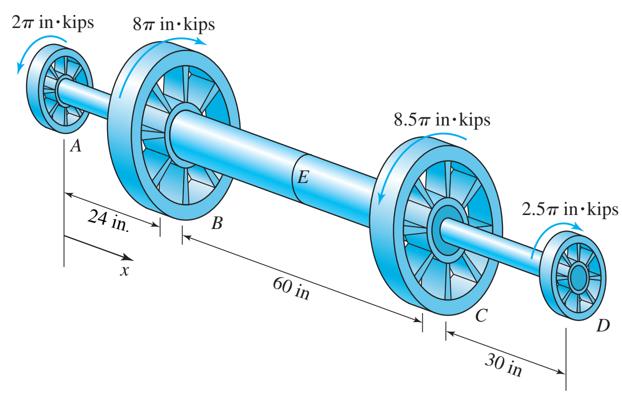


Figure 5.25 Geometry of shaft and loading in Example 5.6.

PLAN

By making imaginary cuts in sections *AB*, *BC*, and *CD* and drawing the free-body diagrams we can find the internal torques in each section. (a) We find the relative rotation in each section using Equation (5.12). Summing the relative rotations we can obtain $\phi_D - \phi_A$. (b) We find the maximum shear stress in each section using Equation (5.10), then by comparison find the maximum shear stress τ_{max} in the shaft. (c) In part (b) we found the shear stress in section *BC*. We obtain the direction of the shear stress either using the subscript or intuitively.

SOLUTION

The polar moment of inertias for each segment can be obtained as

$$J_{AB} = J_{CD} = \frac{\pi}{32}(2 \text{ in.})^4 = \frac{\pi}{2} \text{ in.}^4 \qquad J_{BC} = \frac{\pi}{32}(4 \text{ in.})^4 = 8\pi \text{ in.}^4 \tag{E1}$$

We make an imaginary cuts, draw internal torques as per our sign convention and obtain the free body diagrams as shown in Figure 5.25. We obtain the internal torques in each segment by equilibrium of moment about shaft axis:

$$\boldsymbol{T}_{AB} + 2\pi \text{ in.}\cdot\text{kips} = 0 \qquad \text{or} \qquad \boldsymbol{T}_{AB} = -2\pi \text{ in.}\cdot\text{kips} \tag{E2}$$

$$\boldsymbol{T}_{BC} + 2\pi \text{ in.}\cdot\text{kips} - 8\pi \text{ in.}\cdot\text{kips} = 0 \qquad \text{or} \qquad \boldsymbol{T}_{BC} = 6\pi \text{ in.}\cdot\text{kips} \tag{E3}$$

$$\boldsymbol{T}_{CD} + 2.5\pi \text{ in.}\cdot\text{kips} = 0 \qquad \text{or} \qquad \boldsymbol{T}_{CD} = -2.5\pi \text{ in.}\cdot\text{kips} \tag{E4}$$

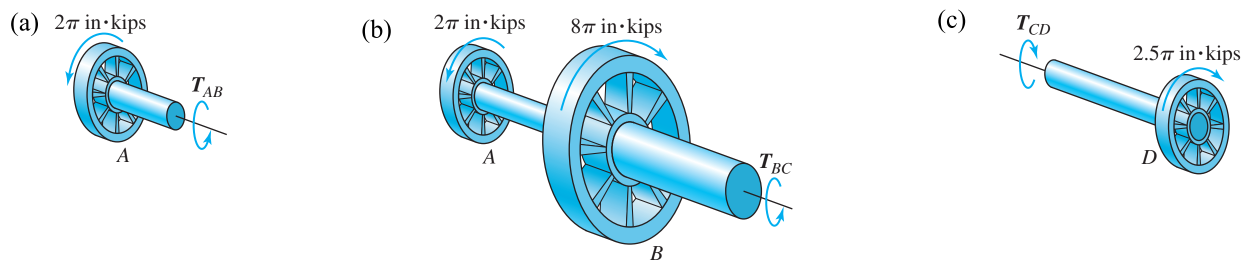


Figure 5.26 Free-body diagrams in Example 5.6 after an imaginary cut in segment (a) *AB,* (b) *BC,* and (c) *CD*.

(a) From Equation (5.12), we obtain the relative rotations of the end of segments as

$$\phi_B - \phi_A = \frac{\boldsymbol{T}_{AB}(x_B - x_A)}{G_{AB}J_{AB}} = \frac{(-2\pi \text{ in.}\cdot\text{kips})(24 \text{ in.})}{(12{,}000 \text{ ksi})(\pi/2 \text{ in.}^4)} = -8(10^{-3}) \text{ rad} \tag{E5}$$

$$\phi_C - \phi_B = \frac{\boldsymbol{T}_{BC}(x_C - x_B)}{G_{BC}J_{BC}} = \frac{(6\pi \text{ in.}\cdot\text{kips})(60 \text{ in.})}{(12{,}000 \text{ ksi})(8\pi \text{ in.}^4)} = 3.75(10^{-3}) \text{ rad} \tag{E6}$$

$$\phi_D - \phi_C = \frac{\boldsymbol{T}_{CD}(x_D - x_C)}{G_{CD}J_{CD}} = \frac{(-2.5\pi \text{ in.}\cdot\text{kips})(30 \text{ in.})}{(12{,}000 \text{ ksi})(\pi/2 \text{ in.}^4)} = -12.5(10^{-3}) \text{ rad} \tag{E7}$$

Adding Equations (E5), (E6), and (E7), we obtain the relative rotation of the section at *D* with respect to the section at *A*:

$$\phi_D - \phi_A = (\phi_B - \phi_A) + (\phi_C - \phi_B) + (\phi_D - \phi_C) = (-8 + 3.75 - 12.5)(10^{-3}) \text{ rad} = -16.75(10^{-3}) \text{ rad} \tag{E8}$$

ANS. $\phi_D - \phi_A = 0.01675$ rad cw

(b) The maximum torsional shear stress in section *AB* and CD will exist at $\rho = 1$ and in BC it will exist at $\rho = 2$. From Equation (5.10) we can obtain the maximum shear stress in each segment:

$$(\tau_{AB})_{\max} = \frac{\boldsymbol{T}_{AB}(\rho_{AB})_{\max}}{J_{AB}} = \frac{(-2\pi \text{ in.}\cdot\text{kips})(1 \text{ in.})}{(\pi/2 \text{ in.}^4)} = -4 \text{ ksi} \tag{E9}$$

$$(\tau_{BC})_{\max} = \frac{\boldsymbol{T}_{BC}(\rho_{BC})_{\max}}{J_{BC}} = \frac{(6\pi \text{ in.}\cdot\text{kips})(2 \text{ in.})}{(8\pi \text{ in.}^4)} = 1.5 \text{ ksi} \tag{E10}$$

$$(\tau_{CD})_{max} = \frac{\boldsymbol{T}_{CD}(\rho_{CD})_{max}}{J_{CD}} = \frac{(-2.5\pi \text{ in.}\cdot\text{kips})(1 \text{ in.})}{(\pi/2 \text{ in.}^4)} = -5 \text{ ksi} \tag{E11}$$

From Equations (E9), (E10), and (E11) we see that the magnitude of maximum torsional shear stress is in segment *CD*.

ANS. $|\tau_{max}| = 5$ ksi

(c) The direction of shear stress at point E can be determined as described below.

Shear stress direction using subscripts: In Figure 5.27*a* we note that $\tau_{x\theta}$ in segment *BC* is +1.5 ksi. The outward normal is in the positive *x* direction and the force has to be pointed in the positive θ direction (tangent direction), which at point *E* is downward.

Shear stress direction determined intuitively: Figure 5.27*b* shows a schematic of segment *BC*. Consider an imaginary section through *E* in segment *BC*. Segment *BE* tends to rotate clockwise with respect to segment *EC*. The shear stress will oppose the imaginary clockwise motion of segment *BE*; hence the direction will be counterclockwise, as shown.

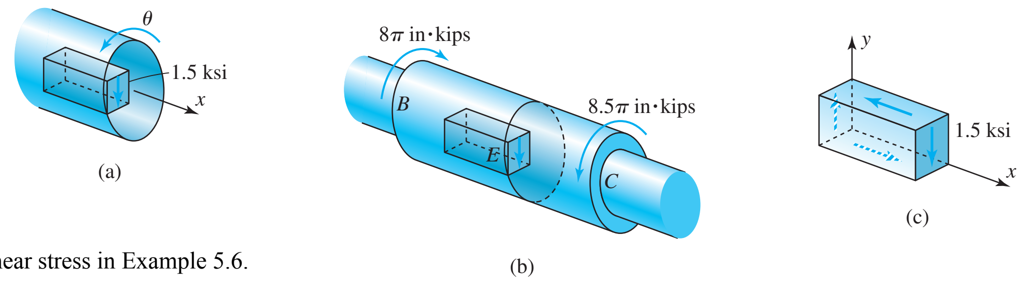


Figure 5.27 Direction of shear stress in Example 5.6.

We complete the rest of the stress cube using the fact that a pair of symmetric shear stresses points either toward the corner or away from the corner, as shown in Figure 5.27*c*.

COMMENTS

1. Suppose that we do not follow the sign convention for internal torque. Instead, we show the internal torque in a direction that counterbalances the external torque as shown in Figure 5.28. Then in the calculation of $\phi_D - \phi_A$ the addition and subtraction must be done manually to account for clockwise and counterclockwise rotation. Also, the shear stress direction must now be determined intuitively.

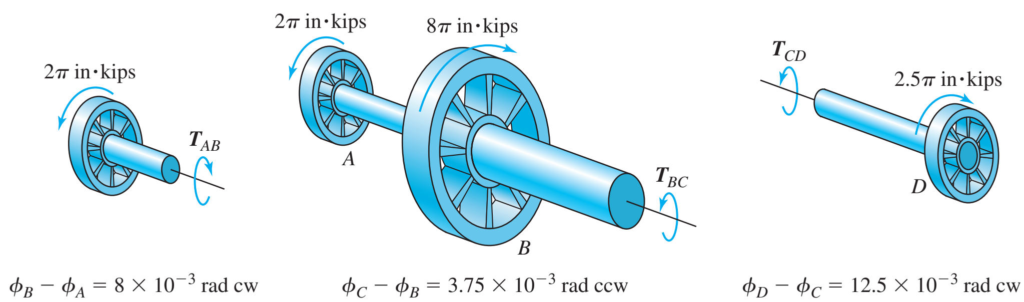


Figure 5.28 Intuitive analysis in Example 5.6.

2. An alternative perspective of the calculation of $\phi_D - \phi_A$ is as follows:

$$\phi_D - \phi_A = \int_{x_A}^{x_D} \frac{\boldsymbol{T}}{GJ}\,dx = \int_{x_A}^{x_B} \frac{\boldsymbol{T}_{AB}}{G_{AB}J_{AB}}\,dx + \int_{x_B}^{x_C} \frac{\boldsymbol{T}_{BC}}{G_{BC}J_{BC}}\,dx + \int_{x_C}^{x_D} \frac{\boldsymbol{T}_{CD}}{G_{CD}J_{CD}}\,dx$$

or, written compactly,

$$\Delta\phi = \sum_i \frac{\boldsymbol{T}_i\,\Delta x_i}{G_i J_i} \tag{5.13}$$

3. Note that $\boldsymbol{T}_{BC} - \boldsymbol{T}_{AB} = 8\pi$ is the magnitude of the applied external torque at the section at *B*. Similarly $\boldsymbol{T}_{CD} - \boldsymbol{T}_{BC} = -8.5\pi$, which is the magnitude of the applied external torque at the section at *C*. In other words, the internal torques jump by the value of the external torque as one crosses the external torque from left to right. We will make use of this observation in the next section when plotting the torque diagram.

5.2.6 Torque Diagram

A **torque diagram** is a plot of the internal torque across the entire shaft. To construct torque diagrams we create a small torsion template to guide us in which direction the internal torque will jump. A **torsion template** is an infinitesimal segment of the shaft constructed by making imaginary cuts on either side of a supposed external torque.

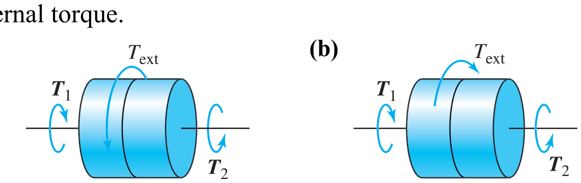


Figure 5.29 Torsion templates and equations.

Template Equations (a) $\boldsymbol{T}_2 = \boldsymbol{T}_1 - T_{\text{ext}}$ (b) $\boldsymbol{T}_2 = \boldsymbol{T}_1 + T_{\text{ext}}$

Figure 5.29 shows torsion templates. The external torque can be drawn either clockwise or counterclockwise. The ends of the torsion templates represent the imaginary cuts just to the left and just to the right of the applied external torque. The internal torques on these cuts are drawn according to the sign convention. An equilibrium equation is written, which we will call the **template equation**

If the external torque on the shaft is in the direction of the assumed torque shown on the template, then the value of $\boldsymbol{T}_2$ is calculated according to the template equation. If the external torque on the shaft is opposite to the direction shown, then $\boldsymbol{T}_2$ is calculated by changing the sign of T_{ext} in the template equation. Moving across the shaft using the template equation, we can then draw the torque diagram, as demonstrated in the next example.

EXAMPLE 5.7

Calculate the rotation of the section at D with respect to the section at A by drawing the torque diagram using the template shown in Figure 5.29.

PLAN

We can start the process by considering an imaginary extension on the left end. In the imaginary extension the internal torque is zero. Using the template in Figure 5.29*a* to guide us, we can draw the torque diagram.

SOLUTION

Let LA be an imaginary extension on the left side of the shaft, as shown in Figure 5.30. Clearly the internal torque in the imaginary section LA is zero, that is, $\boldsymbol{T}_1 = 0$. The torque at A is in the same direction as the torque T_{ext} shown on the template in Figure 5.29*a*. Using the template equation, we subtract the value of the applied torque to obtain a value of -2π in.·kips for the internal torque $\boldsymbol{T}_2$ just after wheel A. This is the starting value in the internal torque diagram.

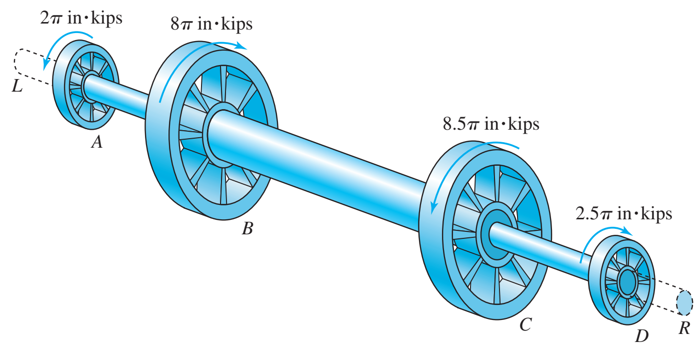


Figure 5.30 Imaginary extensions of the shaft in Example 5.7.

We approach wheel B with an internal torque value of -2π in.·kips, that is, $\boldsymbol{T}_1 = -2\pi$ in.·kips. The torque at B is in the opposite direction to the torque shown on the template in Figure 5.29*a* we add 8π in·kips to obtain a value of $+6\pi$ in.·kips for the internal torque just after wheel B.

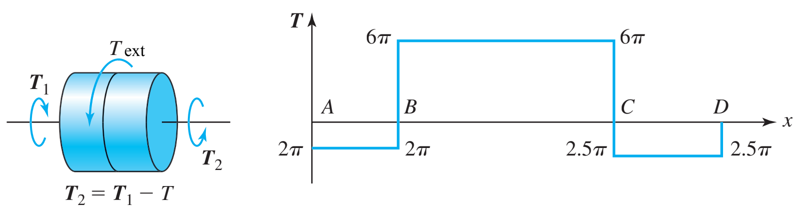


Figure 5.31 Torque diagram in Example 5.7.

We approach wheel C with a value of 6π in.·kips and note that the torque at C is in the same direction as that shown on the template in Figure 5.29*a*. Hence we subtract 8.5π in.·kips as per the template equation to obtain -2.5π in.·kips for the internal torque just after wheel C. The torque at D is in the same direction as that on the template, and on adding we obtain a zero value in the imaginary extended bar DR as expected, for the shaft is in equilibrium.

From Figure 5.31 the internal torque values in the segments are

$$\boldsymbol{T}_{AB} = -2\pi \text{ in.}\cdot\text{kips} \qquad \boldsymbol{T}_{BC} = 6\pi \text{ in.}\cdot\text{kips} \qquad \boldsymbol{T}_{CD} = -2.5\pi \text{ in.}\cdot\text{kips} \tag{E1}$$

To obtain the relative rotation of wheel D with respect to wheel A, we substitute the torque values in Equation (E1) into Equation (5.13):

$$\phi_D - \phi_A = \frac{\boldsymbol{T}_{AB}(x_B - x_A)}{G_{AB}J_{AB}} + \frac{\boldsymbol{T}_{BC}(x_C - x_B)}{G_{BC}J_{BC}} + \frac{\boldsymbol{T}_{CD}(x_D - x_C)}{G_{CD}J_{CD}} \quad \text{or}$$

$$\phi_D - \phi_A = \frac{(-2\pi \text{ in.}\cdot\text{kips})(24 \text{ in.})}{(12{,}000 \text{ ksi})(\pi/2 \text{ in.}^4)} + \frac{(6\pi \text{ in.}\cdot\text{kips})(60 \text{ in.})}{(12{,}000 \text{ ksi})(8\pi \text{ in.}^4)} + \frac{(-2.5\pi \text{ in.}\cdot\text{kips})(30 \text{ in.})}{(12{,}000 \text{ ksi})(\pi/2 \text{ in.}^4)} = 16.75(10^{-3}) \text{ rad} \tag{E2}$$

ANS. $\phi_D - \phi_A = 0.01675$ rad cw

COMMENT

1. We could have created the torque diagram using the template shown in Figure 5.29*b* and the template equation. It may be verified that we obtain the same torque diagram. This shows that the direction of the applied torque T_{ext} on the template is immaterial.

EXAMPLE 5.8

A 1-m-long hollow shaft in Figure 5.32 is to transmit a torque of 400 N·m. The shaft can be made of either titanium alloy or aluminum. The shear modulus of rigidity G, the allowable shear stress τ_{allow}, and the density γ are given in Table 5.1. The outer diameter of the shaft must be 25 mm to fit existing attachments. The relative rotation of the two ends of the shaft is limited to 0.375 rad. Determine the inner radius to the nearest millimeter of the lightest shaft that can be used for transmitting the torque.

TABLE 5.1 **Material properties in Example 5.8**

Material	G (GPa)	τ_{allow} (MPa)	γ (Mg/m^3)
Titanium alloy	36	450	4.4
Aluminum	28	150	2.8

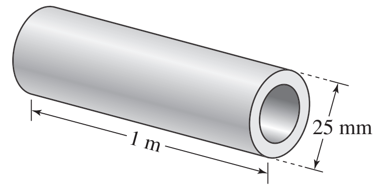


Figure 5.32 Shaft in Example 5.8.

PLAN

The change in inner radius affects only the polar moment J and no other quantity in Equations 5.10 and 5.12. For each material we can find the minimum polar moment J needed to satisfy the stiffness and strength requirements. Knowing the minimum J for each material we can find the maximum inner radius. We can then find the volume and hence the mass of each material and make our decision on the lighter shaft.

SOLUTION

We note that for both materials $\rho_{max} = 0.0125$ m and $x_2 - x_1 = 1$ m. From Equations 5.10 and 5.12 for titanium alloy we obtain limits on J_{Ti} shown below.

$$(\Delta\phi)_{Ti} = \frac{(400\ \text{N}\cdot\text{m})(1\ \text{m})}{[36(10^9)\ \text{N/m}^2]J_{Ti}} \le 0.375\ \text{rad} \qquad \text{or} \qquad J_{Ti} \ge 29.63(10^{-9})\ \text{m}^4 \tag{E1}$$

$$(\tau_{max})_{Ti} = \frac{(400\ \text{N}\cdot\text{m})(0.0125\ \text{m})}{J_{Ti}} \le 450(10^6)\ \text{N/m}^2 \qquad \text{or} \qquad J_{Ti} \ge 11.11(10^{-9})\ m^4 \tag{E2}$$

Using similar calculations for the aluminum shaft we obtain the limits on J_{Al}:

$$(\Delta\phi)_{Al} = \frac{(400\ \text{N}\cdot\text{m})(1\ \text{m})}{[28(10^9)\ \text{N/m}^2]\times J_{Al}} \le 0.375\ \text{rad} \qquad \text{or} \qquad J_{Al} \ge 38.10(10^{-9})\ \text{m}^4 \tag{E3}$$

$$(\tau_{max})_{Al} = \frac{(400\ \text{N}\cdot\text{m})(0.0125\ \text{m})}{J_{Al}} \le 150(10^6)\ \text{N/m}^2 \qquad \text{or} \qquad J_{Al} \ge 33.33(10^{-9})\ \text{m}^4 \tag{E4}$$

Thus if $J_{Ti} \ge 29.63(10^{-9})\ \text{m}^4$, it will meet both conditions in Equations (E1) and (E2). Similarly if $J_{Al} \ge 38.10(10^{-9})\ \text{m}^4$, it will meet both conditions in Equations (E3) and (E4). The internal diameters D_{Ti} and D_{Al} can be found as follows:

$$J_{Ti} = \frac{\pi}{32}(0.025^4 - D_{Ti}^4) \ge 29.63(10^{-9}) \qquad D_{Ti} \le 17.3(10^{-3})\ \text{m} \tag{E5}$$

$$J_{Al} = \frac{\pi}{32}(0.025^4 - D_{Al}^4) \ge 38.10(10^{-9}) \qquad D_{Al} \le 7.1(10^{-3})\ \text{m} \tag{E6}$$

Rounding downward to the closest millimeter, we obtain

$$D_{Ti} = 17(10^{-3})\ \text{m} \qquad D_{Al} = 7(10^{-3})\ \text{m} \tag{E7}$$

We can find the mass of each material from the material density as

$$M_{Ti} = [4.4(10^6)\ \text{g/m}^3]\left[\frac{\pi}{4}(0.025^2 - 0.017^2)\ \text{m}^2\right](1\ \text{m}) = 1161\ g \tag{E8}$$

$$M_{Al} = [2.8(10^6)\ \text{g/m}^3]\left[\frac{\pi}{4}(0.025^2 - 0.007^2)\ \text{m}^2\right](1\ \text{m}) = 1267\ g \tag{E9}$$

From Equations (E8) and (E9) we see that the titanium alloy shaft is lighter.

ANS. A titanium alloy shaft should be used with an inside diameter of 17 mm**.**

COMMENTS

1. For both materials the stiffness limitation dictated the calculation of the internal diameter, as can be seen from Equations (E1) and (E3).
2. Even though the density of aluminum is lower than that titanium alloy, the mass of titanium is less. Because of the higher modulus of rigidity of titanium alloy we can meet the stiffness requirement using less material than for aluminum.

3. If in Equation (E5) we had $17.95(10^{-3})$ m on the right side, our answer for D_{Ti} would still be 17 mm because we have to round downward to ensure meeting the less-than sign requirement in Equation (E5).

EXAMPLE 5.9

The radius of a tapered circular shaft varies from $4r$ units to r units over a length of $40r$ units, as shown in Figure 5.33. The radius of the uniform shaft shown is r units. Determine (a) the angle of twist of wheel C with respect to the fixed end in terms of T, r, and G; (b) the maximum shear stress in the shaft.

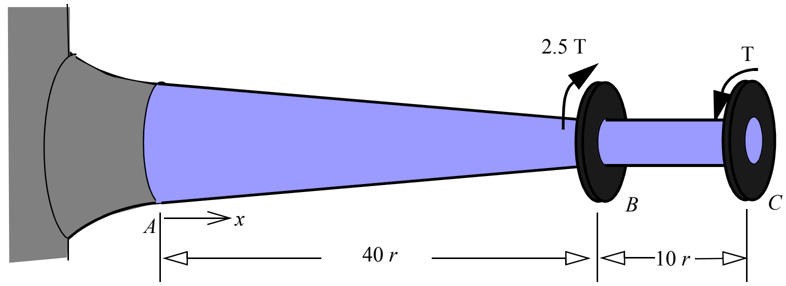


Figure 5.33 Shaft geometry in Example 5.9

PLAN

(a) We can find the relative rotation of wheel C with respect to wheel B using Equation (5.12). For section AB we obtain the polar moment J as a function of x and integrate Equation (5.9) to obtain the relative rotation of B with respect to A. We add the two relative rotations and obtain the relative rotation of C with respect to A. (b) As per Equation (5.10), the maximum shear stress will exist where the shaft radius is minimum (J is minimum) and T is maximum. Thus by inspection, the maximum shear stress will exist on a section just left of B.

SOLUTION

We note that R is a linear function of x and can be written as $R(x) = a + bx$. Noting that at $x = 0$ the radius $R = 4r$ we obtain $a = 4r$. Noting that $x = 40r$ the radius $R = r$ we obtain $b = -3r/(40r) = -0.075$. The radius R can be written as

$$R(x) = 4r - 0.075x \tag{E1}$$

Figure 5.34 shows the free body diagrams after imaginary cuts have been made and internal torques drawn as per our sign convention. By equilibrium of moment about the shaft axis we obtain the internal torques:

$$\boldsymbol{T}_{BC} = T \tag{E2}$$

$$\boldsymbol{T}_{AB} + 2.5T - T = 0 \qquad \text{or} \qquad \boldsymbol{T}_{AB} = -1.5T \tag{E3}$$

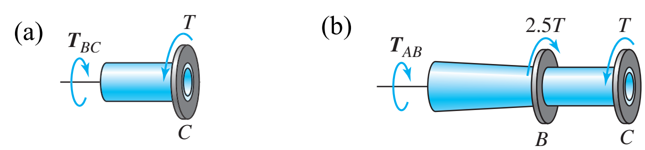


Figure 5.34 Free-body diagrams in Example 5.9 after imaginary cut in segment (a) BC (b) AB

The polar moment of inertias can be written as

$$J_{BC} = \frac{\pi}{2}r^4 \qquad J_{AB} = \frac{\pi}{2}R^4 = \frac{\pi}{2}(4r - 0.075x)^4 \tag{E4}$$

(a) We can find the relative rotation of the section at C with respect to the section at B using Equation (5.12):

$$\phi_C - \phi_B = \frac{\boldsymbol{T}_{BC}(x_C - x_B)}{G_{BC}J_{BC}} = \frac{T(10r)}{G(\pi/2)r^4} = \frac{6.366T}{Gr^3} \tag{E5}$$

Substituting Equations (E3) and (E4) into Equation (5.9) and integrating from point A to point B, we can find the relative rotation at the section at B with respect to the section at A:

$$\left(\frac{d\phi}{dx}\right)_{AB} = \frac{\boldsymbol{T}_{AB}}{G_{AB}J_{AB}} = \frac{-1.5T}{G(\pi/2)(4r - 0.075x)^4} \qquad \text{or} \qquad \int_{\phi_A}^{\phi_B} d\phi = \int_{x_A}^{x_B} -\frac{3T}{G\pi(4r - 0.075x)^4}\,dx \quad \text{or}$$

$$\phi_B - \phi_A = -\frac{3T}{G\pi}\frac{1}{-3}\frac{1}{-0.075}\frac{1}{(4r - 0.075x)^3}\Bigg|_0^{40r} = -\frac{T}{0.075G\pi}\left[\frac{1}{r^3} - \frac{1}{(4r)^3}\right] = -4.178\frac{T}{Gr^3} \tag{E6}$$

Adding Equations (E5) and (E6), we obtain

$$\phi_C - \phi_A = \frac{T}{Gr^3}(6.366 - 4.178) \tag{E7}$$

ANS. $\phi_C - \phi_A = 2.2\dfrac{T}{Gr^3}$ ccw

(b) Just left of the section at B we have $J_{AB} = \pi r^4/2$ and $\rho_{max} = r$. Substituting these values into Equation (5.10), we obtain the maximum torsional shear stress in the shaft as

$$\tau_{max} = \frac{-1.5Tr}{\pi r^4/2} = -\frac{0.955T}{r^3} \tag{E8}$$

ANS. $|\tau_{max}| = 0.955(T/r^3)$

Dimension check: The dimensional consistency[3] of the answer is checked as follows:

$$T \rightarrow O(FL) \qquad r \rightarrow O(L) \qquad G \rightarrow O\left(\frac{F}{L^2}\right) \qquad \phi \rightarrow O(\,) \qquad \frac{T}{Gr^3} \rightarrow O\left(\frac{FL}{\frac{F}{L^2}L^3}\right) \rightarrow O(\,) \rightarrow \text{checks}$$

$$\tau \rightarrow O\left(\frac{F}{L^2}\right) \qquad \frac{T}{r^3} \rightarrow O\left(\frac{FL}{L^3}\right) \rightarrow O\left(\frac{F}{L^2}\right) \rightarrow \text{checks}$$

COMMENT

1. The direction of the shear stress can be determined using subscripts or intuitively, as shown in Figure 5.35.

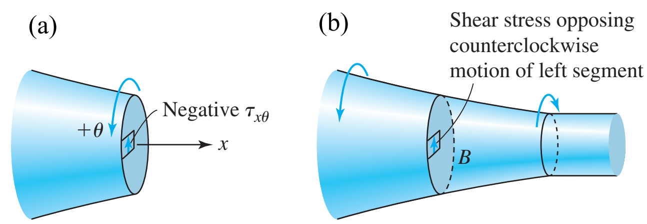


Figure 5.35 Direction of shear stress in Example 5.9: (a) by subscripts; (b) by inspection.

EXAMPLE 5.10

A uniformly distributed torque of q in.·lb/in. is applied to an entire shaft, as shown in Figure 5.36. In addition to the distributed torque a concentrated torque of $T = 3qL$ in.·lb is applied at section B. Let the shear modulus be G and the radius of the shaft r. In terms of q, L, G, and r, determine: (a) The rotation of the section at C. (b) The maximum shear stress in the shaft.

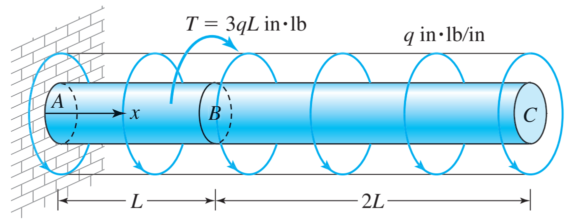


Figure 5.36 Shaft and loading in Example 5.10.

PLAN

(a) The internal torque in segments AB and BC as a function of x must be determined first. Then the relative rotation in each section is found by integrating Equation (5.9). (b) Since J and ρ_{max} are constant over the entire shaft, the maximum shear stress will exist on a section where the internal torque is maximum. By plotting the internal torque as a function of x we can determine its maximum value.

Figure 5.37 shows the free body diagrams after imaginary cuts are made at x distance from A and internal torques drawn as per our sign convention. We replace the distributed torque by an equivalent torque that is equal to the distributed torque intensity multiplied by the length of the cut shaft (the rectangular area). From equilibrium of moment about the shaft axis in Figure 5.37 we obtain the internal torques:

$$\boldsymbol{T}_{AB} + 3qL - q(3L - x) = 0 \qquad \text{or} \qquad \boldsymbol{T}_{AB} = -qx \tag{E1}$$

$$\boldsymbol{T}_{BC} - q(3L - x) = 0 \qquad \text{or} \qquad \boldsymbol{T}_{BC} = q(3L - x) \tag{E2}$$

[3] $O(\,)$ represents the dimension of the quantity on the left. F represents dimension for the force. L represents the dimension for length. Thus shear modulus, which has dimension of force (F) per unit area (L^2), is represented as $O(F/L^2)$.

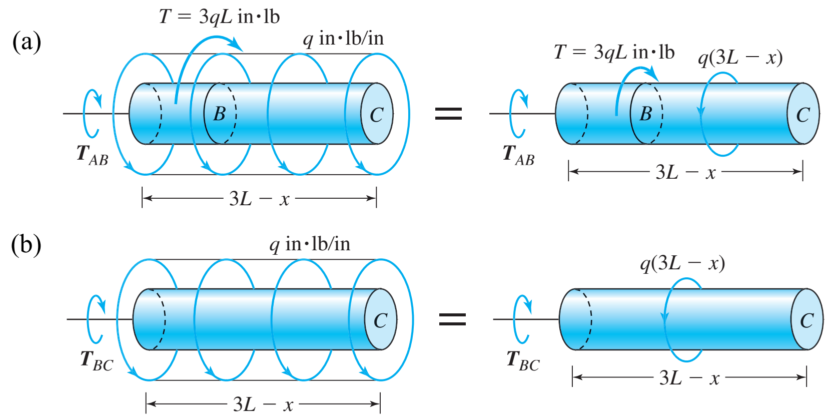


Figure 5.37 Free-body diagrams in Example 5.10 after imaginary cut in segment (a) *AB*, and (b) *BC*.

Integrating Equation (5.9) for each segment we obtain the relative rotations of segment ends as

$$\left(\frac{d\phi}{dx}\right)_{AB} = \frac{T_{AB}}{G_{AB}J_{AB}} = \frac{-qx}{G\pi r^4/2} \quad\text{or}\quad \int_{\phi_A}^{\phi_B} d\phi = -\int_{x_A=0}^{x_B=L}\frac{2qx}{G\pi r^4}\,dx \quad\text{or}\quad \phi_B-\phi_A = -\frac{qx^2}{G\pi r^4}\bigg|_0^L = -\frac{qL^2}{G\pi r^4} \tag{E3}$$

$$\left(\frac{d\phi}{dx}\right)_{BC} = \frac{T_{BC}}{G_{BC}J_{BC}} = \frac{q(3L-x)}{G\pi r^4/2} \quad\text{or}\quad \int_{\phi_B}^{\phi_C} d\phi = \int_{x_B=L}^{x_C=3L}\frac{2q(3L-x)}{G\pi r^4}\,dx \quad\text{or}$$

$$\phi_C-\phi_B = \frac{2q}{G\pi r^4}\left(3Lx-\frac{x^2}{2}\right)\bigg|_L^{3L} = \frac{2q}{G\pi r^4}\left[9L^2-\frac{(3L)^2}{2}-3L^2+\frac{L^2}{2}\right] = \frac{4qL^2}{G\pi r^4} \tag{E4}$$

(a) Adding Equations (E3) and (E4), we obtain the rotation of the section at *C* with respect to the section at *A*:

$$\phi_C-\phi_A = -\frac{qL^2}{G\pi r^4}+\frac{4qL^2}{G\pi r^4} \tag{E5}$$

ANS. $\phi_C-\phi_A = = \left(\frac{3qL^2}{G\pi r^4}\right)\text{ccw}$

(b) Figure 5.38 shows the plot of the internal torque as a function of *x* using Equations (E1) and (E2). The maximum torque will occur on a section just to the right of B. From Equation (5.10) the maximum torsional shear stress is

$$\tau_{max} = \frac{T_{max}\rho_{max}}{J} = \frac{(2qL)(r)}{\pi r^4/2} \tag{E6}$$

ANS. $\tau_{max} = \frac{4qL}{\pi r^3}$

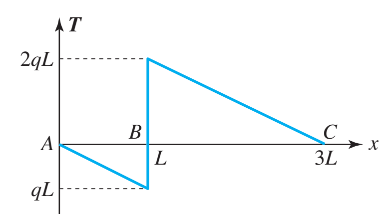


Figure 5.38 Torque diagram in Example 5.10.

Dimension check: The dimensional consistency (see footnote 12) of our answers is checked as follows:

$$q \to O\left(\frac{FL}{L}\right) \to O(F) \qquad r\to O(L) \qquad L\to O(L) \qquad G\to O\left(\frac{F}{L^2}\right)$$

$$\phi\to O(\;) \qquad \frac{qL^2}{Gr^4}\to O\left(\frac{FL^2}{(F/L^2)L^4}\right)\to O(\;)\to\text{checks} \qquad \tau\to O\left(\frac{F}{L^2}\right) \qquad \frac{qL}{r^3}\to O\left(\frac{FL}{L^3}\right)\to O\left(\frac{F}{L^2}\right)\to\text{checks}$$

COMMENT

1. A common mistake is to write the incorrect length of the shaft as a function of *x* in the free-body diagrams. It should be remembered that the location of the cut is defined by the variable *x*, which is measured from the common origin for all segments. Each cut produces two parts, and we are free to choose either part.

5.2.7* General Approach to Distributed Torque

Distributed torques are usually due to inertial forces or frictional forces acting on the surface of the shaft. The internal torque $\boldsymbol{T}$ becomes a function of x when a shaft is subjected to a distributed external torque, as seen in Example 5.10. If $t(x)$ is a simple function, then we can find $\boldsymbol{T}$ as a function of x by drawing a free-body diagram, as we did in Example 5.10. However, if the distributed torque $t(x)$ is a complex function (see Problems 5.39, 5.61, and 5.62), it may be easier to use the alternative solution method described in this section.

Consider an infinitesimal shaft element that is created by making two imaginary cuts at a distance dx from each other, as shown in Figure 5.39*a*.

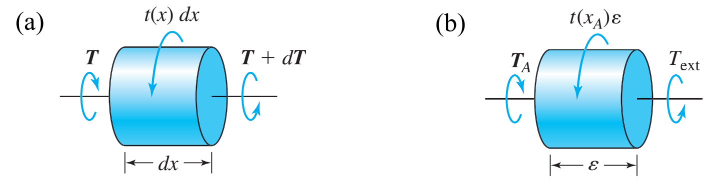


Figure 5.39 (a) Equilibrium of an infinitesimal shaft element. (b) Boundary condition on internal torque.

By equilibrium moments about the axis of the shaft, we obtain $(\boldsymbol{T} + d\boldsymbol{T}) + t(x)\ dx - \boldsymbol{T} = 0$ or

$$\boxed{\frac{d\boldsymbol{T}}{dx} + t(x) = 0} \tag{5.14}$$

Equation (5.14) represents the equilibrium equation at any section x. It assumes that $t(x)$ is positive counterclockwise with respect to the x axis. The sign of $\boldsymbol{T}$ obtained from Equation (5.14) corresponds to the direction defined by the sign convention. If $t(x)$ is zero in a segment of a shaft, then the internal torque is constant in that segment.

Equation (5.14) can be integrated to obtain the internal torque $\boldsymbol{T}$. The integration constant can be found by knowing the value of the internal torque $\boldsymbol{T}$ at either end of the shaft. To obtain the value of $\boldsymbol{T}$ at the end of the shaft (say, point A), a free-body diagram is constructed after making an imaginary cut at an infinitesimal distance ε from the end, as shown in Figure 5.39*b*.We then write the equilibrium equation as

$$\lim_{\varepsilon \to 0}[T_{\text{ext}} - \boldsymbol{T}_A - t(x_A)\varepsilon] = 0 \qquad \text{or} \qquad \boldsymbol{T}_A = T_{\text{ext}} \tag{5.15}$$

Equation (5.15) shows that the distributed torque does not affect the boundary condition on the internal torque. The value of the internal torque $\boldsymbol{T}$ at the end of the shaft is equal to the concentrated external torque applied at the end. Equation (5.14) is a differential equation. Equation (5.15) is a boundary condition. A differential equation and all the conditions necessary to solve it is called the **boundary value problem.**

EXAMPLE 5.11

The external torque on a drill bit varies linearly to a maximum intensity of q in.·lb/in., as shown in Figure 5.40. If the drill bit diameter is d, its length L, and the modulus of rigidity G, determine the relative rotation of the end of the drill bit with respect to the chuck.

PLAN

The relative rotation of section B with respect to section A has to be found. We can substitute the given distributed torque in Equation (5.14) and integrate to find the internal torque as a function of x. We can find the integration constant by using the condition that at section B the internal torque will be zero. We can substitute the internal torque expression into Equation (5.9) and integrate from point A to point B to find the relative rotation of section B with respect to section A.

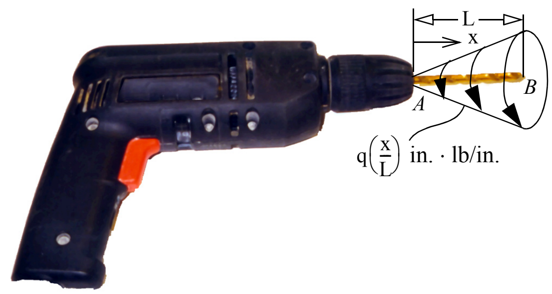


Figure 5.40 Distributed torque on a drill bit in Example 5.11.

SOLUTION

The distributed torque on the drill bit is counterclockwise with respect to the x axis. Thus we can substitute $t(x) = q(x/L)$ into Equation (5.14) to obtain the differential equation shown as Equation (E1). At point B, that is, at $x = L$, the internal torque should be zero as there is no concentrated applied torque at B.The boundary condition is shown as Equation (E2). The boundary value problem statement is

- Differential Equation

$$\frac{d\boldsymbol{T}}{dx} + q\frac{x}{L} = 0 \tag{E1}$$

- Boundary Condition

$$\boldsymbol{T}(x = L) = 0 \tag{E2}$$

Integrating Equation (E1) we obtain

$$\boldsymbol{T} = -q\frac{x^2}{2L} + c \tag{E3}$$

Substituting Equation (E2) into Equation (E3) we obtain the integration constant c as

$$-q\frac{L^2}{2L} + c = 0 \qquad \text{or} \qquad c = \frac{qL}{2} \tag{E4}$$

Substituting Equation (E4) into Equation (E3) we obtain internal torque as

$$T = \frac{q}{2L}(L^2 - x^2) \tag{E5}$$

Substituting Equation (E5) into Equation (5.9) and integrating we obtain the relative rotation of the section at B with respect to the section at A as

$$\frac{d\phi}{dx} = \frac{(q/2L)(L^2 - x^2)}{G\pi d^4/32} \qquad \text{or} \qquad \int_{\phi_A}^{\phi_B} d\phi = \frac{16q}{\pi G L d^4}\int_{x_A=0}^{x_B=L}(L^2 - x^2)\,dx \qquad \text{or} \qquad \phi_B - \phi_A = \frac{16q}{\pi G L d^4}\left(L^2 x - \frac{x^3}{3}\right)\Big|_0^L \tag{E6}$$

ANS. $\phi_B - \phi_A = \dfrac{32qL^2}{3\pi G d^4}$ ccw

Dimension check: The dimensional consistency (see footnote 12) of our answer is checked as follows:

$$q \rightarrow O\left(\frac{FL}{L}\right) \rightarrow O(F) \qquad d \rightarrow O(L) \qquad L \rightarrow O(L) \qquad G \rightarrow O\left(\frac{F}{L^2}\right) \qquad \phi \rightarrow O(\,) \qquad \frac{qL^2}{Gd^4} \rightarrow O\left(\frac{FL^2}{(F/L^2)L^4}\right) \rightarrow O(\,) \rightarrow \text{checks}$$

COMMENTS

1. No free-body diagram was needed to find the internal torque because Equation (5.14) is an equilibrium equation. It is therefore valid at each and every section of the shaft.
2. We could have obtained the internal torque by integrating Equation (5.14) from L to x as follows:

$$\int_{T_B=0}^{T} d\boldsymbol{T} = -\int_{x_B=L}^{x} t(x)\,dx = -\int_{L}^{x} q\left(\frac{x}{L}\right)dx = \frac{q}{2L}(L^2 - x^2)$$

3. The internal torque can also be found using a free-body diagram. We can make an imaginary cut at some location x and draw the free-body diagram of the right side. The distributed torque represented by $\int_x^L t(x)\,dx$ is the area of the trapezoid $BCDE$, and this observation can be used in drawing a statically equivalent diagram, as shown in Figure 5.41. Equilibrium then gives us the value of the internal torque as before. We can find the internal torque as shown.

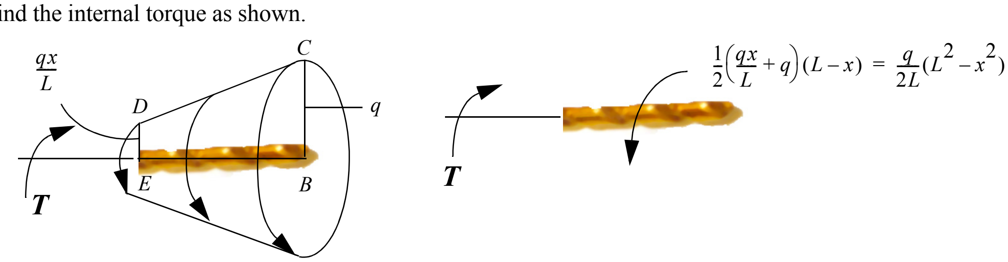


Figure 5.41 Internal torque by free-body diagram in Example 5.11.

4. The free-body diagram approach in Figure 5.41 is intuitive but more tedious and difficult than the use of Equation (5.14). As the function representing the distributed torque grows in complexity, the attractiveness of the mathematical approach of Equation (5.14) grows correspondingly.

MoM in Action: Drill, the Incredible Tool

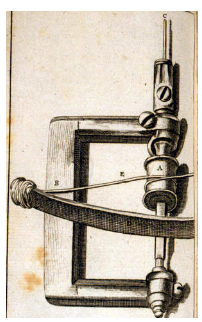
Drills have been in use for almost as long as humans have used tools. Early humans knew from experience that friction generated by torquing a wooden shaft could start a fire—a technique still taught in survivalist camps. Archeologists in Pakistan have found teeth perhaps 9000 years old showing the concentric marks of a flint stone drill. The Chinese used larger drills in the 3rd B.C.E. to extract water and oil from earth. The basic design—a chuck that delivers torque to the drill bit—has not changed, but their myriad uses to make holes from the very small to the very large continues to grow.

Early development of the drill was driven by the technology of delivering power to the drill bit. In 1728, French dentist Pierre Fauchard (Figure 5.42*a*) described how catgut twisted around a cylinder could power the rotary movement as a bow moved back and forth. However, hand drills like these operated at only about 15 rpm. George F. Harrington introduced the first motor-driven drill in 1864, powered by the spring action of a clock. George Green, an American dentist, introduced a pedal-operated pneumatic drill just four years later—and, in 1875, an electric drill. By 1914 dental drills could operate at 3000 rpm.

Other improvements took better understanding of the relationship between power, torsion, and shear stress in the drill bit (problems 5.45—5.47) and the material being drilled:

- The sharper the drill tip, the higher the shear stresses at the point, and the greater the amount of material that can be sheared. For most household jobs the angle of the drill tip is 118°. For soft materials such as plastic, the angle is sharper, while for harder material such as steel the angle is shallower.
- For harder materials low speeds can prolong the life of drill bit. However, in dentistry higher speeds, of up to 500,000 rpm, reduce a patient's pain.

(a)



(b)

Figure 5.42 (a) Pierre Fauchard drill. (b) Tunnel boring machine Matilda (Courtesy Erikt9).

- Slower speeds are also used to shear a large amount of material. Tunnel boring machines (TBM) shown in Figure 5.42*b* may operate at 1 to 10 rpm. The world's largest TBM, with a diameter of 14.2 m, was used to drill the Elbe Tunnel in Hamburg, Germany. Eleven TBM's drilled the three pipes of the English Channel, removing 10.5 million cubic yards of earth in seven years.
- Drill bits can be made of steel, tungsten carbide, polycrystalline diamonds, titanium nitrate, and diamond powder. The choice is dictated by the material to be drilled as well as the cost. Even household drills have different bits for wood, metal, or masonry.

Delivery and control of power to the drill bit are engineering challenges. So is removal of sheared material, not only to prevent the hole from plugging, but also because the material carries away heat, improving the strength and life of a drill bit. Yet the fundamental function of a drill remains: shearing through torsion.

PROBLEM SET 5.2

5.19 The torsional shear stress at point A on a solid circular homogenous cross-section was found to be τ_A= 120 MPa. Determine the maximum torsional shear stress on the cross-section.

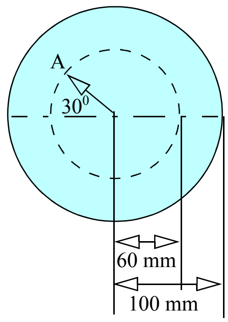


Figure P5.19

5.20 The torsional shear strain at point *A* on a homogenous circular section shown in Figure P5.20 was found to be 900 μ rads. Using a shear modulus of elasticity of 4000 ksi, determine the torsional shear stress at point *B*.

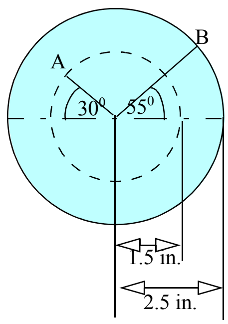
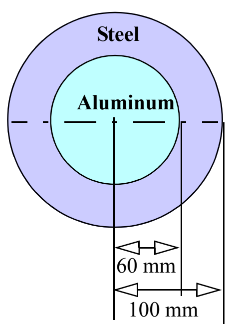


Figure P5.20

5.21 An aluminum shaft (G_{al}= 28 GPa) and a steel shaft (G_S=82 GPa) are securely fastened to form composite shaft with a cross section shown in Figure P5.21. If the maximum torsional shear strain in aluminum is 1500 μ rads, determine the maximum torsional shear strain in steel.



FigureP5.21

5.22 An aluminum shaft (G_{al}= 28 GPa) and a steel shaft (G_S=82 GPa) are securely fastened to form composite shaft with a cross section shown in Figure P5.21. If the maximum torsional shear stress in aluminum is 21 MPa, determine the maximum torsional shear stress in steel.

5.23 Determine the direction of torsional shear stress at points *A* and *B* in Figure P5.23 (a) by inspection; (b) by using the sign convention for internal torque and the subscripts. Report your answer as a positive or negative τ_{xy}.

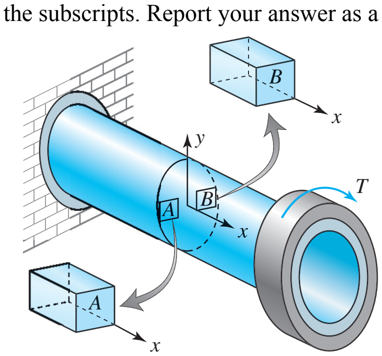


Figure P5.23

5.24 Determine the direction of torsional shear stress at points A and B in Figure P5.24 (a) by inspection; (b) by using the sign convention for internal torque and the subscripts. Report your answer as a positive or negative τ_{xy}

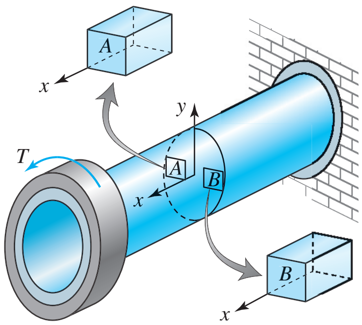


Figure P5.24

5.25 Determine the direction of torsional shear stress at points A and B in Figure P5.25 (a) by inspection; (b) by using the sign convention for internal torque and the subscripts. Report your answer as a positive or negative τ_{xy}.

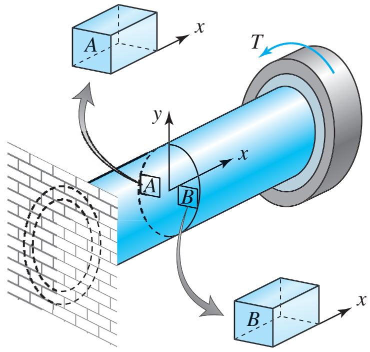


Figure P5.25

5.26 Determine the direction of torsional shear stress at points A and B in Figure P5.26 (a) by inspection; (b) by using the sign convention for internal torque and the subscripts. Report your answer as a positive or negative τ_{xy}.

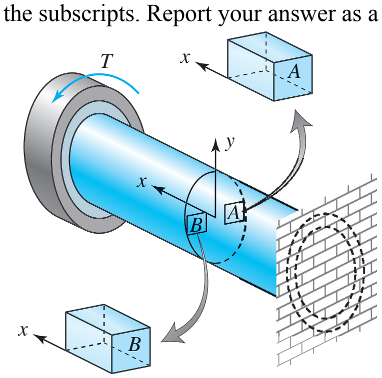
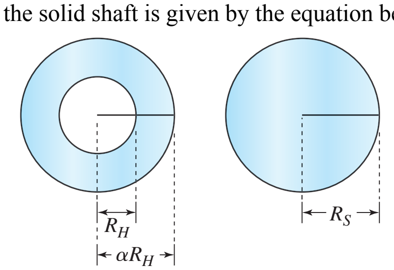


Figure P5.26

5.27 The two shafts shown in Figure P5.27 have the same cross sectional areas A. Show that the ratio of the polar moment of inertia of the hollow shaft to that of the solid shaft is given by the equation below.:



$$\frac{J_{\text{hollow}}}{J_{\text{solid}}} = \frac{\alpha^2 + 1}{\alpha^2 - 1}$$

Figure P5.27

5.28 Show that for a thin tube of thickness t and center-line radius R the polar moment of inertia can be approximated by $J = 2\pi R^3 t$. By thin tube we imply $t < R/10$.

5.29 (a) Draw the torque diagram in Figure P5.29. (b) Check the values of internal torque by making imaginary cuts and drawing free-body diagrams. (c) Determine the rotation of the rigid wheel D with respect to the rigid wheel A if the torsional rigidity of the shaft is 90,000 kips·in.^{2}.

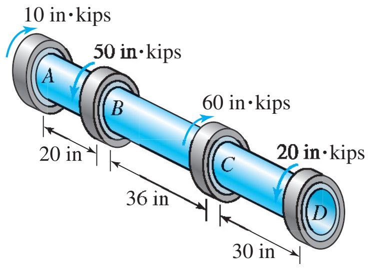


Figure P5.29

5.30 (a) Draw the torque diagram in Figure P5.30. (b) Check the values of internal torque by making imaginary cuts and drawing free-body diagrams. (c) Determine the rotation of the rigid wheel *D* with respect to the rigid wheel *A* if the torsional rigidity of the shaft is 1270 $kN \cdot m^2$.

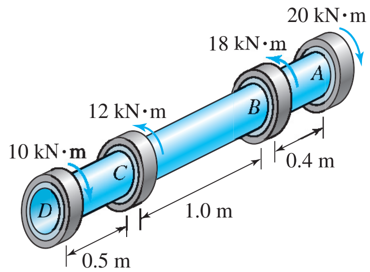


Figure P5.30

5.31 The shaft in Figure P5.31 is made of steel ($G = 80$ GPa) and has a diameter of 150 mm. Determine (a) the rotation of the rigid wheel *D*; (b) the magnitude of the torsional shear stress at point *E* and show it on a stress cube (Point *E* is on the top surface of the shaft.); (c) the magnitude of maximum torsional shear *strain* in the shaft.

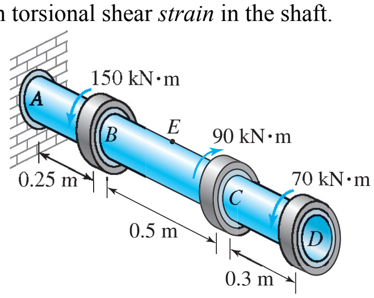


Figure P5.31

5.32 The shaft in Figure P5.32 is made of aluminum ($G = 4000$ ksi) and has a diameter of 4 in. Determine (a) the rotation of the rigid wheel *D*; (b) the magnitude of the torsional shear stress at point *E* and show it on a stress cube (Point *E* is on the bottom surface of the shaft.); (c) the magnitude of maximum torsional shear *strain* in the shaft.

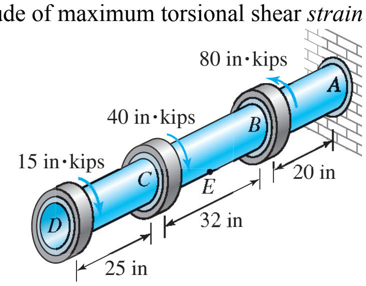


Figure P5.32

5.33 Two circular steel shafts (G =12,000 ksi) of diameter 2 in. are securely connected to an aluminum shaft (G =4,000 ksi) of diameter 1.5 in. as shown in Figure P5.33. Determine (a) the rotation of section at D with respect to the wall, and (b) the maximum shear stress in the shaft.

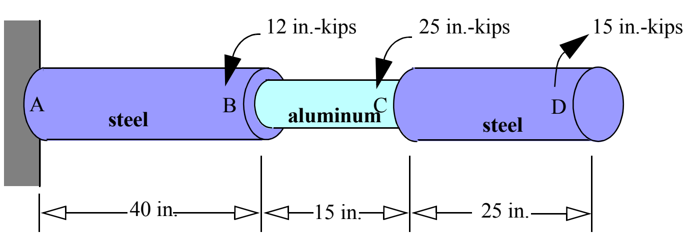


Figure P5.33

5.34 A solid circular steel shaft *BC* (G_s = 12,000 ksi) is securely attached to two hollow steel shafts *AB* and *CD*, as shown in Figure P5.34. Determine (a) the angle of rotation of the section at *D* with respect to the section at *A*; (b) the magnitude of maximum torsional shear stress in the shaft; (c) the torsional shear stress at point *E* and show it on a stress cube. (Point *E* is on the inside bottom surface of *CD*.)

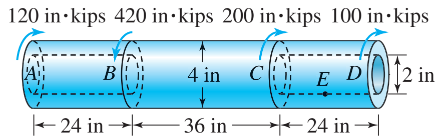


Figure P5.34

5.35 A steel shaft (G = 80 GPa) is subjected to the torques shown in Figure P5.35. Determine (a) the rotation of section *A* with respect to the no-load position; (b) the torsional shear stress at point *E* and show it on a stress cube. (Point *E* is on the surface of the shaft.)

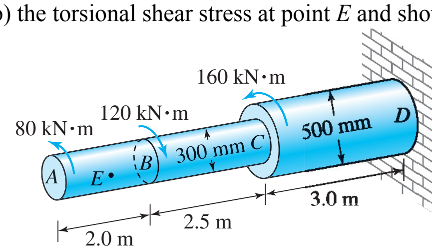


Figure P5.35

Tapered shafts

5.36 The radius of the tapered circular shaft shown in Figure P5.36 varies from 200 mm at *A* to 50 mm at *B*. The shaft between *B* and *C* has a constant radius of 50 mm. The shear modulus of the material is G = 40 GPa. Determine (a) the angle of rotation of wheel *C* with respect to the fixed end; (b) the maximum shear strain in the shaft

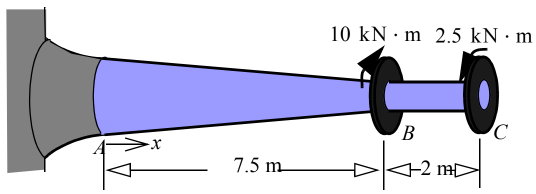


Figure P5.36

5.37 The radius of the tapered shaft in Figure P5.37 varies as $R = Ke^{-ax}$. Determine the rotation of the section at *B* in terms of the applied torque T_{ext}, length L, shear modulus of elasticity G, and geometric parameters K and a.

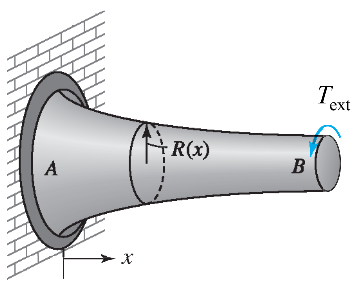


Figure P5.37

5.38 The radius of the tapered shaft shown in Figure P5.37 varies as $R = r\sqrt{(2 - 0.25x/L)}$. In terms of T_{ext}, L, G, and r, determine (a) the rotation of the section at *B*; (b) the magnitude of maximum torsional shear stress in the shaft.

Distributed torques

5.39 The external torque on a drill bit varies as a quadratic function to a maximum intensity of q in.·lb/in., as shown in Figure P5.39. If the drill bit diameter is d, its length L, and its modulus of rigidity G, determine (a) the maximum torsional shear stress on the drill bit; (b) the relative rotation of the end of the drill bit with respect to the chuck.

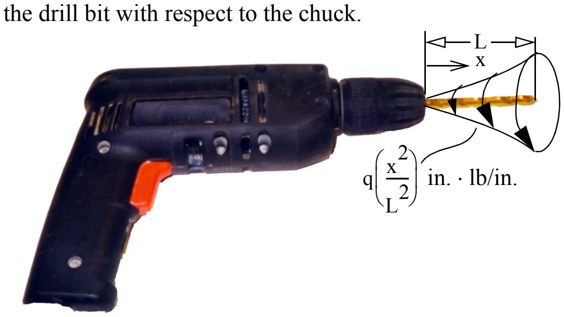


Figure P5.39

5.40 A circular solid shaft is acted upon by torques, as shown in Figure P5.40. Determine the rotation of the rigid wheel A with respect to the fixed end C in terms of q, L, G, and J.

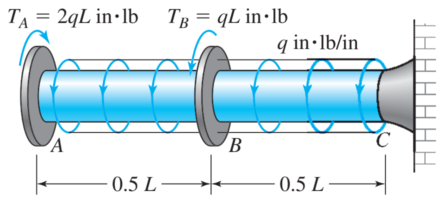


Figure P5.40

Design problems

5.41 A thin steel tube ($G = 12{,}000$ ksi) of $\frac{1}{8}$-in. thickness has a mean diameter of 6 in. and a length of 36 in. What is the maximum torque the tube can transmit if the allowable torsional shear stress is 10 ksi and the allowable relative rotation of the two ends is 0.015 rad?

5.42 Determine the maximum torque that can be applied on a 2-in. diameter solid aluminum shaft ($G = 4000$ ksi) if the allowable torsional shear stress is 18 ksi and the relative rotation over 4 ft of the shaft is to be limited to 0.2 rad.

5.43 A hollow steel shaft ($G = 80$ GPa) with an outside radius of 30 mm is to transmit a torque of 2700 N·m. The allowable torsional shear stress is 120 MPa and the allowable relative rotation over 1 m is 0.1 rad. Determine the maximum permissible inner radius to the nearest millimeter.

5.44 A 5-ft-long hollow shaft is to transmit a torque of 200 in.·kips. The outer diameter of the shaft must be 6 in. to fit existing attachments. The relative rotation of the two ends of the shaft is limited to 0.05 rad. The shaft can be made of steel or aluminum. The shear modulus of elasticity G, the allowable shear stress τ_{allow}, and the specific weight γ are given in Table P5.44. Determine the maximum inner diameter to the nearest $\frac{1}{8}$ in. of the lightest shaft that can be used for transmitting the torque and the corresponding weight.

TABLE P5.44

Material	G (ksi)	τ_{allow} (ksi)	γ (lb/in.^{3})
Steel	12,000	18	0.285
Aluminum	4,000	10	0.100

Transmission of power

Power P is the rate at which work dW/dt is done; and work W done by a constant torque is equal to the product of torque T and angle of rotation ϕ. Noting that $\omega = d\phi/dt$, we obtain

$$P = T\omega = 2\pi f T \tag{5.16}$$

where T is the torque transmitted, ω is the rotational speed in radians per second, and f is the frequency of rotation in hertz (Hz). Power is reported in units of horsepower in U.S. customary units or in watts. 1 horsepower (hp) is equal to 550 ft·lb/s = 6600 in·lb/s and 1 watt (W) is equal to 1 N·m/s. Use Equation (5.16) to solve Problems 5.38 through 5.40.

5.45 A 100-hp motor is driving a pulley and belt system, as shown in Figure P5.45. If the system is to operate at 3600 rpm, determine the minimum diameter of the solid shaft AB to the nearest $\frac{1}{8}$ in. if the allowable stress in the shaft is 10 ksi.

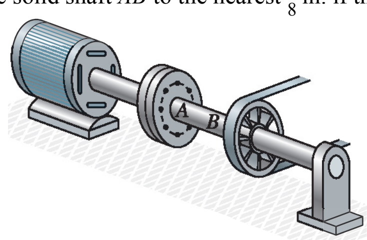


Figure P5.45

5.46 The bolts used in the coupling for transferring power in Problem 5.45 have an allowable strength of 12 ksi. Determine the minimum number (> 4) of $\frac{1}{4}$-in. diameter bolts that must be placed at a radius of $\frac{5}{8}$ in.

5.47 A 20-kW motor drives three gears, which are rotating at a frequency of 20 Hz. Gear A next to the motor transfers 8 kW of power. Gear B, which is in the middle, transfers 7 kW of power. Gear C, which is at the far end from the motor, transfers the remaining 5 kW of power. A single solid steel shaft connecting the motors to all three gears is to be used. The steel used has a yield strength in shear of 145

MPa. Assuming a factor of safety of 1.5, what is the minimum diameter of the shaft to the nearest millimeter that can be used if failure due to yielding is to be avoided? What is the magnitude of maximum torsional stress in the segment between gears *A* and *B*?

Stretch yourself

5.48 A circular shaft has a constant torsional rigidity *GJ* and is acted upon by a distributed torque *t*(*x*). If at section *A* the internal torque is zero, show that the relative rotation of the section at *B* with respect to the rotation of the section at *A* is given by

$$\phi_B - \phi_A = \frac{1}{GJ}\left[\int_{x_A}^{x_B}(x - x_B)t(x)\,dx\right] \tag{5.17}$$

5.49 A composite shaft made from *n* materials is shown in Figure P5.49. G_i and J_i are the shear modulus of elasticity and polar moment of inertia of the i[th] material. (a) If Assumptions from 1 through 6 are valid, show that the stress $(\tau_{x\theta})_i$ in the i[th] material is given Equation (5.18a), where $\boldsymbol{T}$ is the total internal torque at a cross section. (b) If Assumptions 8 through 10 are valid, show that relative rotation $\phi_2 - \phi_1$ is given by Equation (5.18b). (c) Show that for $G_1 = G_2 = G_3 .... = G_n = G$ Equations (5.18a) and (5.18b) give the same results as Equations (5.10) and (5.12).

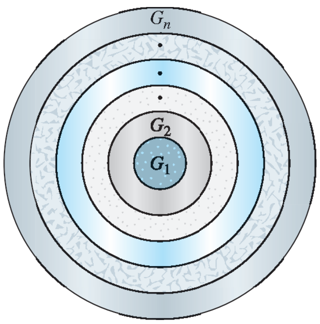


Figure P5.49

$$(\tau_{x\theta})_i = \frac{G_i \rho \boldsymbol{T}}{\sum_{j=1}^{n} G_j J_j} \tag{5.18a}$$

$$\phi_2 - \phi_1 = \frac{\boldsymbol{T}(x_2 - x_1)}{\sum_{j=1}^{n} G_j J_j} \tag{5.18b}$$

5.50 A circular solid shaft of radius *R* is made from a nonlinear material that has a shear stress-shear strain relationship given by $\tau = G\gamma^{0.5}$. Assume that the kinematic assumptions are valid and shear strain varies linearly with the radial distance across the cross-section. Determine the maximum shear stress and the rotation of section at B in terms of external torque T_{ext}, radius *R*, material constant *G*, and length *L*.

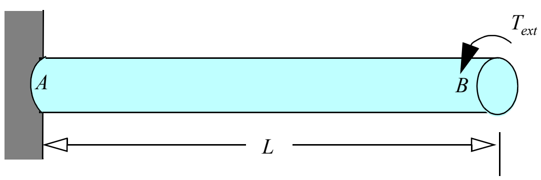


Figure P5.50

5.51 A hollow circular shaft is made from a non-linear materials that has the following shear stress--shear strain relation $\tau = G\gamma^2$. Assume that the kinematic assumptions are valid and shear strain varies linearly with the radial distance across the cross-section. In terms of internal torque $\boldsymbol{T}$, material constant *G*, and *R*, obtain formulas for (a) the maximum shear stress τ_{max} and (b) the relative rotation $\phi_2 - \phi_1$ of two cross-sections at x_1 and x_2.

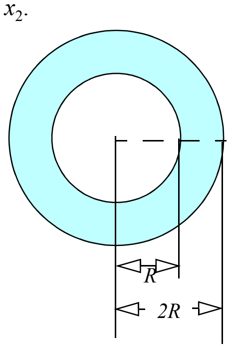


Figure P5.51

5.52 A solid circular shaft of radius *R* and length *L* is twisted by an applied torque *T*. The stress–strain relationship for a nonlinear material is given by the power law $\tau = G\gamma^n$. If Assumptions 1 through 4 are applicable, show that the maximum shear stress in the shaft and the relative rotation of the two ends are as follows:

$$\tau_{\max} = \frac{\boldsymbol{T}(n+3)}{2\pi R^3} \qquad \Delta\phi = \left[\frac{(n+3)\boldsymbol{T}}{2\pi G R^{3+n}}\right]^{1/n} L$$

Substitute $n = 1$ in the formulas and show that we obtain the same results as from Equations 5.10 and 5.12.

5.53 The internal torque $\boldsymbol{T}$ and the displacements of a point on a cross section of a noncircular shaft shown in Figure P5.53 are given by the equations below

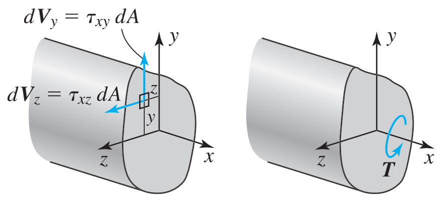


Figure P5.53 Torsion of noncircular shafts.

$$u = \psi(y, z)\frac{d\phi}{dx} \qquad (5.19a)$$

$$v = -xz\frac{d\phi}{dx} \qquad (5.19b)$$

$$w = xy\frac{d\phi}{dx} \qquad (5.19c)$$

$$\boldsymbol{T} = \int_A (y\tau_{xz} - z\tau_{xy})\, dA \qquad (5.20)$$

where u, v, and w are the displacements in the x, y, and z directions, respectively; $d\phi/dx$ is the rate of twist and is considered constant. $\psi(x, y)$ is called the *warping function*[4] and describes the movement of points out of the plane of cross section. Using Equations (2.12d) and (2.12f) and Hooke's law, show that the shear stresses for a noncircular shaft are given by

$$\tau_{xy} = G\left(\frac{\partial\psi}{\partial y} - z\right)\frac{d\phi}{dx} \qquad \tau_{xz} = G\left(\frac{\partial\psi}{\partial z} + y\right)\frac{d\phi}{dx} \qquad (5.21)$$

5.54 Show that for circular shafts, $\psi(x, y) = 0$, the equations in Problem 5.53 reduce to Equation (5.9).

5.55 Consider the dynamic equilibrium of the differential element shown in Figure P5.55, where $\boldsymbol{T}$ is the internal torque, γ is the material density, J is the polar area moment of inertia, and $\partial^2\phi/\partial t^2$ is the angular acceleration. Show that dynamic equilibrium results in Equation (5.22)

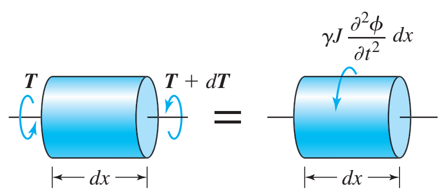


Figure P5.55 Dynamic equilibrium.

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2} \quad \text{where } c = \sqrt{\frac{G}{\gamma}} \qquad (5.22)$$

5.56 Show by substitution that the solution of Equation (5.23) satisfies Equation (5.22):

$$\phi = \left(A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}\right) \times (C\cos\omega t + D\sin\omega t) \qquad (5.23)$$

where A, B, C, and D are constants that are determined from the boundary conditions and the initial conditions and ω is the frequency of vibration.

Computer problems

5.57 A hollow aluminum shaft of 5 ft in length is to carry a torque of 200 in.·kips. The inner radius of the shaft is 1 in. If the maximum torsional shear stress in the shaft is to be limited to 10 ksi, determine the minimum outer radius to the nearest $\frac{1}{8}$ in.

5.58 A 4-ft-long hollow shaft is to transmit a torque of 100 in.·kips. The relative rotation of the two ends of the shaft is limited to 0.06 rad. The shaft can be made of steel or aluminum. The shear modulus of rigidity G, the allowable shear stress τ_{allow}, and the specific weight γ are given in Table P5.58. The inner radius of the shaft is 1 in. Determine the outer radius of the lightest shaft that can be used for transmitting the torque and the corresponding weight.

TABLE P5.58

Material	G (ksi)	τ_{allow} (ksi)	γ (lb/in.^{3})
Steel	12,000	18	0.285
Aluminum	4000	10	0.100

5.59 Table P5.59 shows the measured radii of the solid tapered shaft shown in Figure P5.59, at several points along the axis of the shaft. The shaft is made of aluminum (G = 28 GPa) and has a length of 1.5 m. Determine: (a) the rotation of the free end with respect to the wall using numerical integration; (b) the maximum shear stress in the shaft.

[4]Equations of elasticity show that the warping function satisfies the Laplace equation, $\partial^2\psi/\partial y^2 + \partial^2\psi/\partial z^2 = 0$.

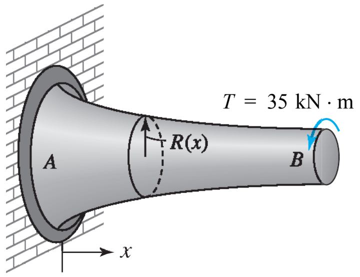


Figure P5.59

TABLE P5.59

x (m)	R(x) (mm)
0.0	100.6
0.1	92.7
0.2	82.6
0.3	79.6
0.4	75.9
0.5	68.8
0.6	68.0
0.7	65.9
0.8	60.1
0.9	60.3
1.0	59.1
1.1	54.0
1.2	54.8
1.3	54.1
1.4	49.4
1.5	50.6

5.60 Let the radius of the tapered shaft in Problem 5.59 be represented by the equation $R(x) = a + bx$. Using the data in Table P5.59 determine the constants a and b by the least-squares method and then find the rotation of the section at B by analytical integration.

5.61 Table P5.61 shows the values of distributed torque at several points along the axis of the solid steel shaft (G = 12,000 ksi) shown in Figure P5.61. The shaft has a length of 36 in. and a diameter of 1 in. Determine (a) the rotation of end A with respect to the wall using numerical integration; (b) the maximum shear stress in the shaft.

TABLE P5.61

x (in.)	t(x) (in.·lb/in.)
0	93.0
3	146.0
6	214.1
9	260.4
12	335.0
15	424.7
18	492.6
21	588.8
24	700.1
27	789.6
30	907.4
33	1040.3
36	1151.4

5.62 Let the distributed torque $t(x)$ in Problem 5.61 be represented by the equation $t(x) = a + bx + cx^2$. Using the data in Table P5.61 determine the constants a, b, and c by the least-squares method and then find the rotation of the section at B by analytical integration.

QUICK TEST 5.1

Time: 20 minutes/Total: 20 points

Answer true or false and justify each answer in one sentence. Grade yourself with the answers given in Appendix E.

1. Torsional shear strain varies linearly across a homogeneous cross section.

2. Torsional shear *strain* is a maximum at the outermost radius for a homogeneous and a nonhomogeneous cross section.

3. Torsional shear *stress* is a maximum at the outermost radius for a homogeneous and a nonhomogeneous cross section.

4. The formula $\tau_{x\theta} = \boldsymbol{T}\rho/J$ can be used to find the shear stress on a cross section of a tapered shaft.

5. The formula $\phi_2 - \phi_1 = \boldsymbol{T}(x_2 - x_1)/GJ$ can be used to find the relative rotation of a segment of a tapered shaft.

6. The formula $\tau_{x\theta} = \boldsymbol{T}\rho/J$ can be used to find the shear stress on a cross section of a shaft subjected to distributed torques.

7. The formula $\phi_2 - \phi_1 = \boldsymbol{T}(x_2 - x_1)/GJ$ can be used to find the relative rotation of a segment of a shaft subjected to distributed torques.

8. The equation $\boldsymbol{T} = \int_A \rho\tau_{x\theta}\, dA$ cannot be used for nonlinear materials.

9. The equation $\boldsymbol{T} = \int_A \rho\tau_{x\theta}\, dA$ can be used for a nonhomogeneous cross section.

10. Internal torques jump by the value of the concentrated external torque at a section.

5.3 STATICALLY INDETERMINATE SHAFTS

In Chapter 4 we saw the solution of statically indeterminate axial problems require equilibrium equations and compatibility equations. This is equally true for statically indeterminate shafts. The primary focus in this section will be on the solution of statically indeterminate shafts that are on a single axis. However, equilibrium equations and compatibility equations can also be used for solution of shafts with composite cross section, as will be demonstrated in Example 5.14. The use of equilibrium equations and compatibility equations to shafts on multiple axis is left as exercises in Problem Set 5.3.

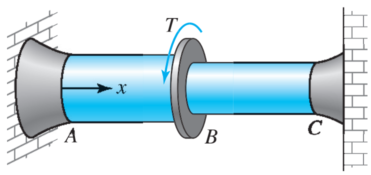


Figure 5.43 Statically indeterminate shaft.

Figure 5.43 shows a statically indeterminate shaft. In statically indeterminate shafts we have two reaction torques, one at the left and the other at the right end of the shaft. But we have only one static equilibrium equation, the sum of all torques in the x direction should be zero. Thus the degree of static redundancy is 1 and we need to generate 1 compatibility equation. We shall use the continuity of ϕ and the fact that the sections at the left and right walls have zero rotation. The compatibility equation state that the relative rotation of the right wall with respect to the left wall is zero. Once more we can use either the displacement method or the force method:

1. In the *displacement method*, we can use the rotation of the sections as the unknowns. If torque is applied at several sections along the shaft, then the rotation of each of the sections is treated as an unknown.
2. In the *force method*, we can use either the reaction torque as the unknown or the internal torques in the sections as the unknowns. Since the degree of static redundancy is 1, the simplest approach is often to take the left wall (or the right wall) reaction as the unknown variable. We can then apply the compatibility equation, as outlined next.

5.3.1 General Procedure for Statically Indeterminate Shafts.

Step 1 Make an imaginary cut in each segment and draw free-body diagrams by taking the left (or right) part if the left (right) wall reaction is carried as the unknown in the problem. Alternatively, draw the torque diagram in terms of the reaction torque.

Step 2 Write the internal torque in terms of the reaction torque.

Step 3 Using Equation (5.12) write the relative rotation of each segment ends in terms of the reaction torque.

Step 4 Add all the relative rotations. Obtain the rotation of the right wall with respect to the left wall and set it equal to zero to obtain the reaction torque.

Step 5 The internal torques can be found from equations obtained in Step 2, and angle of rotation and stresses calculated.

EXAMPLE 5.12

A solid circular steel shaft (G_s = 12,000 ksi, E_s = 30,000 ksi) of 4-in. diameter is loaded as shown in Figure 5.62. Determine the maximum shear stress in the shaft.

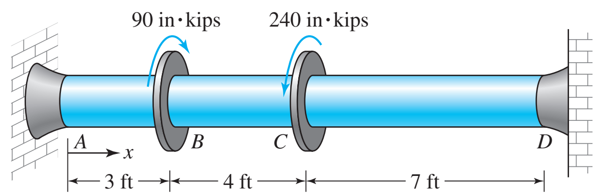


Figure P5.62 Shaft in Example 5.12.

PLAN

We follow the procedure outlined in Section 5.3.1 to determine the reaction torque T_A. For the uniform cross-section the maximum shear stress will occur in the segment that has the maximum internal torque.

SOLUTION

The polar moment of inertia and the torsional rigidity for the shaft can be found as

$$J = \frac{\pi(4\text{ in.})^2}{32} = 25.13\ \text{in.}^4 \qquad GJ = (12000\text{ ksi})(25.13\text{ in.}^4) = 301.6(10^3)\ \text{kips}\cdot\text{in.}^2 \tag{E1}$$

Step 1: We draw the reaction torques T_A and T_D as shown in Figure 5.44*a*. By making imaginary cuts in sections *AB*, *BC*, and *CD* and taking the left part we obtain the free body diagrams shown in Figures 5.44 *b*, *c*, and *d*.

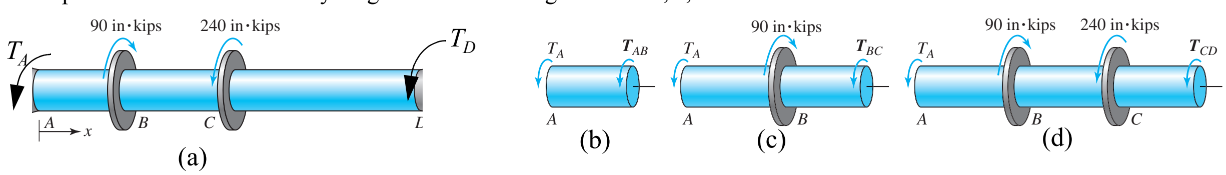


Figure 5.44 Free body diagrams of (a) entire shaft; (b) section *AB*; (c) section *BC*; (d) section *CD*.

Step 2: By equilibrium of moments in Figures 5.44 *b*, *c*, and *d*. or from Figure 5.45*b* we obtain the internal torques as

$$\boldsymbol{T}_{AB} = -T_A \qquad \boldsymbol{T}_{BC} = (-T_A + 90\ )\text{in.}\cdot\text{kips} \qquad \boldsymbol{T}_{CD} = (-T_A - 150\ )\text{in.}\cdot\text{kips} \tag{E2}$$

Step 3: Using Equation (5.12), the relative rotation in each segment ends can be written as

$$\phi_B - \phi_A = \frac{\boldsymbol{T}_{AB}(x_B - x_A)}{G_{AB}J_{AB}} = \frac{-T_A(36)}{301.6(10^3)} = -0.1194(10^{-3})T_A \tag{E3}$$

$$\phi_C - \phi_B = \frac{\boldsymbol{T}_{BC}(x_C - x_B)}{G_{BC}J_{BC}} = \frac{(-T_A + 90)48}{301.6 \times 10^3} = (-0.1592T_A + 14.32)(10^{-3}) \tag{E4}$$

$$\phi_D - \phi_C = \frac{\boldsymbol{T}_{CD}(x_D - x_C)}{G_{CD}J_{CD}} = \frac{(-T_A - 150)84}{301.6 \times 10^3} = (-0.2785T_A - 41.78)(10^{-3}) \tag{E5}$$

Step 4: We obtain $\phi_D - \phi_A$. by adding Equations (E3), (E4), and (E5), which we equate to zero to obtain T_A:

$$\phi_D - \phi_A = (-0.1194T_A - 0.1592T_A + 14.32 - 0.2785T_A - 41.78) = 0 \text{ or}$$

$$T_A = \frac{14.32 - 41.78}{0.1194 + 0.1592 + 0.2785} = -49.28\ \text{in.}\cdot\text{kips} \tag{E6}$$

Step 5: We obtain the internal torques by substituting Equation (E6) into Equation (E2):

$$\boldsymbol{T}_{AB} = 49.28\ \text{in.}\cdot\text{kips} \qquad \boldsymbol{T}_{BC} = 139.28\ \text{in.}\cdot\text{kips} \qquad \boldsymbol{T}_{CD} = -100.72\ \text{in.}\cdot\text{kips} \tag{E7}$$

For the uniform cross-section, the maximum shear stress will occur in segment *BC* and can be found using Equation (5.10):

$$\tau_{max} = \frac{T_{BC}(\rho_{BC})_{max}}{J_{BC}} = \frac{(139.3\ \text{in.}\cdot\text{kips})(2\text{ in.})}{25.13\ \text{in.}^4} \tag{E8}$$

ANS. $\tau_{\max} = 11.1\ \text{ksi}$

COMMENTS

1. We could have found the internal torques in terms of T_A using the template shown in Figure 5.45*a* and drawing the torque diagram in Figure 5.45*b*.

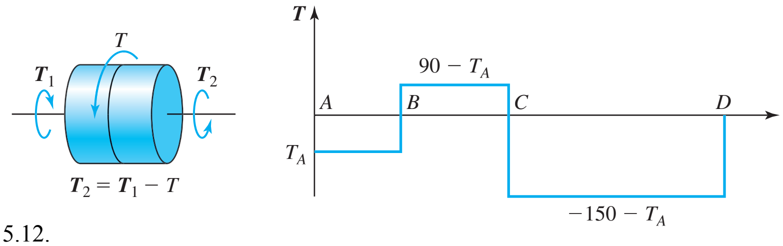


Figure 5.45 Template and torque diagram in Example 5.12.

2. We can find the reaction torque at D from equilibrium of moment in the free body diagram shown in Figure 5.44*d* as:

$$T_D = 90 - 240 - T_A = -60.72 \text{ in.}\cdot\text{kips}$$

3. Because the applied torque at C is bigger than that at B, the reaction torques at A and D will be opposite in direction to the torque at C. In other words, the reaction torques at A and D should by clockwise with respect to the x axis. The sign of T_A and T_D confirms this intuitive reasoning.

EXAMPLE 5.13

A solid aluminum shaft ($G_{al} = 27$ GPa) and a solid bronze shaft ($G_{br} = 45$ GPa) are securely connected to a rigid wheel, as shown in Figure 5.46. The shaft has a diameter of 75 mm. The allowable shear stresses in aluminum and bronze are 100 MPa and 120 MPa, respectively. Determine the maximum torque that can be applied to wheel B.

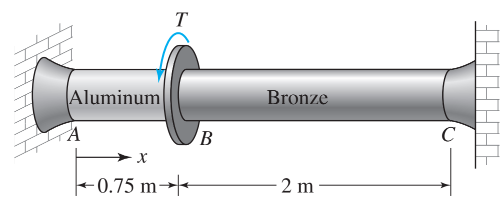


Figure 5.46 Shaft in Example 5.13.

PLAN

We will follow the procedure of Section 5.3.1 and solve for the maximum shear stress in aluminum and bronze in terms of T. We will obtain the two limiting values on T to meet the limitations on maximum shear stress and determine the maximum permissible value of T.

SOLUTION

We can find the polar moment of inertia and the torsional rigidities as

$$J = \pi(0.075 \text{ m})^4/32 = 3.106 \times 10^{-6} \text{ m}^4 \qquad G_{AB}J_{AB} = 83.87(10^3) \text{ N}\cdot\text{m}^2 \qquad G_{BC}J_{BC} = 139.8(10^3) \text{ N}\cdot\text{m}^2 \tag{E1}$$

Step 1: Let T_A, the reaction torque at A, be clockwise with respect to the x axis. We can make imaginary cuts in AB and BC and draw the free-body diagrams as shown in Figure 5.47.

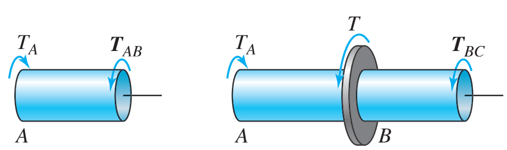


Figure 5.47 Free-body diagrams in Example 5.13.

Step 2: From equilibrium of moment about shaft axis in Figure 5.47 we obtain the internal torques in terms of T_A and T.

$$\boldsymbol{T}_{AB} = T_A \qquad \boldsymbol{T}_{BC} = T_A - T \tag{E2}$$

Step 3: Using Equation (5.12), we obtain the relative rotation in each segment ends as

$$\phi_B - \phi_A = \frac{\boldsymbol{T}_{AB}(x_B - x_A)}{G_{AB}J_{AB}} = \frac{T_A(0.75)}{83.87(10^3)} = 8.942(10^{-6})T_A \tag{E3}$$

$$\phi_C - \phi_B = \frac{\boldsymbol{T}_{BC}(x_C - x_B)}{G_{BC}J_{BC}} = \frac{(T_A - T)(2)}{139.8(10^3)} = (14.31T_A - 14.31T)(10^{-6}) \tag{E4}$$

Step 4: We obtain $\phi_C - \phi_A$ by adding Equations (E3) and (E4) and equate it to zero to find T_A in terms of T:

$$\phi_C - \phi_A = (8.942T_A + 14.31T_A - 14.31T)(10^{-6}) = 0 \qquad \text{or} \qquad T_A = \frac{14.31T}{8.942 + 14.31} = 0.6154T \tag{E5}$$

Step 5: We obtain the internal torques in terms of T by substituting Equation (E5) into Equation (E2):

$$\boldsymbol{T}_{AB} = 0.6154T \qquad \boldsymbol{T}_{BC} = -0.3846T \tag{E6}$$

The maximum shear stress in segment AB and BC can be found in terms of T using Equation (5.10) and noting that $\rho_{\max} = 0.0375$ mm. Using the limits on shear stress we obtain the limits on T as

$$|(\tau_{AB})_{max}| = \left|\frac{T_{AB}(\rho_{AB})_{max}}{J_{AB}}\right| = \frac{(0.6154T)(0.0375\text{ m})}{3.106(10^{-6})\text{ m}^4} = 100(10^6)\text{ N/m}^2 \quad \text{or} \quad T \le 13.46(10^3)\text{ N}\cdot\text{m} \tag{E7}$$

$$|(\tau_{BC})_{\max}| = \left|\frac{T_{BC}(\rho_{BC})_{\max}}{J_{BC}}\right| = \frac{(0.3846T)(0.0375\text{ m})}{3.106(10^{-6})\text{ m}^4} = 120(10^6)\text{ N/m}^2 \quad \text{or} \quad T \le 25.84(10^3)\text{ N}\cdot\text{m} \tag{E8}$$

The value of T that satisfies Equations (E7) and (E8) is the maximum value we seek.

ANS. $T_{\max} = 13.4\text{ kN}\cdot\text{m}$.

COMMENTS

1. The maximum torque is limited by the maximum shear stress in bronze. If we had a limitation on the rotation of the wheel, then we could easily incorporate it by calculating ϕ_B from Equation (E3) in terms of T.
2. We could have solved this problem by the displacement method. In that case we would carry the rotation of the wheel ϕ_B as the unknown.
3. We could have solved the problem by initially assuming that one of the materials reaches its limiting stress value, say aluminum. We can then do our calculations and find the maximum stress in bronze, which would exceed the limiting value of 120 MPa. We would then resolve the problem. The process, though correct, can become tedious as the number of limitations increases. Instead put off deciding which limitation dictates the maximum value of the torque toward the end. In this way we need to solve the problem only once, irrespective of the number of limitations.

EXAMPLE 5.14

A solid steel (G = 80 GPa) shaft is securely fastened to a hollow bronze (G = 40 GPa) shaft as shown in Figure 5.48. Determine the maximum value of shear stress in the shaft and the rotation of the right end with respect to the wall.

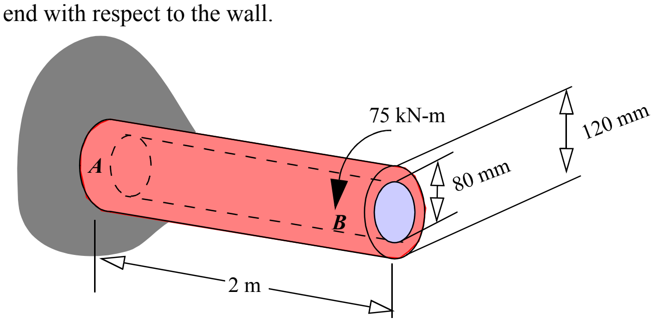


Figure 5.48 Composite shaft in Example 5.14.

PLAN

The steel shaft and the bronze shaft can be viewed as two independent shafts. At equilibrium the sum of the internal torques on each material is equal to the applied torque. The compatibility equation follows from the condition that a radial line on steel and bronze will rotate by the same amount. Hence, the relative rotation is the same for each length segment. Solving the equilibrium equation and the compatibility equation we obtain the internal torques in each material, from which the desired quantities can be found.

SOLUTION

We can find the polar moments and torsional rigidities as

$$J_S = \frac{\pi}{32}(0.08\text{ m})^4 = 4.02(10^{-6})\text{ m}^4 \qquad J_{Br} = \frac{\pi}{32}[(0.12\text{ m})^4 - (0.08\text{ m})^4] = 16.33(10^{-6})\text{ m}^4 \tag{E1}$$

$$G_S J_S = 321.6(10^3)\ \text{N}\cdot\text{m}^2 \qquad G_{Br} J_{Br} = 653.2(10^3)\ \text{N}\cdot\text{m}^2 \tag{E2}$$

Figure 5.49a shows the free body diagram after making an imaginary cut in *AB*. Figure 5.49b shows the decomposition of a composite shaft as two homogenous shafts.

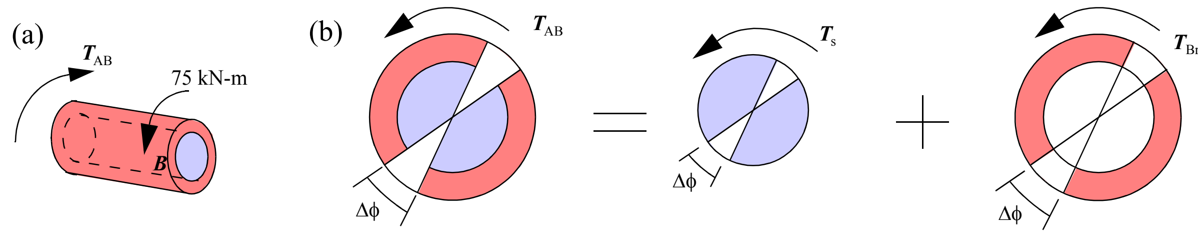


Figure 5.49 (a) Free body diagram (b) Composite shaft as two homogenous shafts in Example 5.14.

From Figure 5.49 we obtain the equilibrium equation,

$$T_{AB} = T_s + T_{Br} = 75\ \text{kN}\cdot\text{m} = 75(10^3)\ \text{N}\cdot\text{m} \tag{E3}$$

Using Equation (5.12) we can write the relative rotation of section at *B* with respect to *A* for the two material as

$$\Delta\phi = \phi_B - \phi_A = \frac{T_s(x_B - x_A)}{G_s J_s} = \frac{T_s(2)}{321.6(10^3)} = 6.219(10^{-6})T_s\ \text{rad} \tag{E4}$$

$$\Delta\phi = \phi_B - \phi_A = \frac{T_{Br}(2)}{653.2(10^3)} = 3.0619(10^{-6})T_{Br}\ \text{rad} \tag{E5}$$

Equating Equations (E4) and (E5) we obtain

$$T_s = 2.03T_{Br} \tag{E6}$$

Solving Equations (E3) and (E4) for the internal torques give

$$T_s = 24.75(10^3)\ \text{N}\cdot\text{m} \qquad T_{Br} = 50.25(10^3)\ \text{N}\cdot\text{m} \tag{E7}$$

Substituting Equation (7) into Equation (4), we find

$$\phi_B - \phi_A = 6.219(10^{-6})(24.75)(10^3) = 0.1538\ \text{rad} \tag{E8}$$

ANS. $\phi_B - \phi_A = 0.1538$ rad ccw

The maximum torsional shear stress in each material can be found using Equation (5.10):

$$(\tau_s)_{max} = \frac{T_s(\rho_s)_{max}}{J_s} = \frac{[24.75(10^3)\ \text{N}\cdot\text{m}](0.04\ \text{m})}{4.02(10^{-6})\ \text{m}^4} = 246.3(10^6)\ \text{N/m}^2 \tag{E9}$$

$$(\tau_{Br})_{max} = \frac{T_{Br}(\rho_{Br})_{max}}{J_{Br}} = \frac{[50.25(10^3)\ \text{N}\cdot\text{m}](0.06\ \text{m})}{16.33(10^{-6})\ \text{m}^4} = 184.6(10^6)\ \text{N/m}^2 \tag{E10}$$

The maximum torsional shear stress is the larger of the two.

ANS. $\tau_{max} = 246.3$ MPa

COMMENT

1. The kinematic condition that all radial lines must rotate by equal amount for a circular shaft had to be explicitly enforced to obtain Equations (E6). We could also have implicitly assumed this kinematic condition and developed formulas for composite shafts (see Problem 5.49) as we did for homogenous shaft. We can then use these formulas to solve statically determinate and indeterminate problems (see Problem 5.82) as we have done for homogenous shafts.

PROBLEM SET 5.3

Statically indeterminate shafts

5.63 A steel shaft (G_{st} = 12,000 ksi) and a bronze shaft (G_{br} = 5600 ksi) are securely connected at B, as shown in Figure P5.63. Determine the maximum torsional shear stress in the shaft and the rotation of the section at *B* if the applied torque *T* = 50 in.·kips.

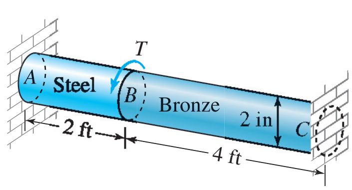


Figure P5.63

5.64 A steel shaft (G_{st} = 12,000 ksi) and a bronze shaft (G_{br} = 5600 ksi) are securely connected at B, as shown in Figure P5.63. Determine the maximum torsional shear strain and the applied torque T if the section at B rotates by an amount of 0.02 rad.

5.65 Two hollow aluminum shafts (G = 10,000 ksi) are securely fastened to a solid aluminum shaft and loaded as shown Figure P5.65. If T = 300 in.·kips, determine (a) the rotation of the section at C with respect to the wall at A; (b) the shear strain at point E. Point E is on the inner surface of the shaft.

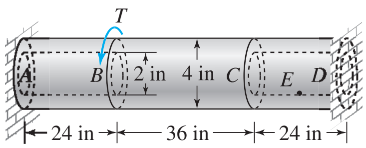


Figure P5.65

5.66 Two hollow aluminum shafts (G = 10,000 ksi) are securely fastened to a solid aluminum shaft and loaded as shown Figure P5.65. The torsional shear strain at point E which is on the inner surface of the shaft is –250 μ. Determine the rotation of the section at C and the applied torque T that produced this shear strain.

5.67 Two solid circular steel shafts (G_{st} = 80 GPa) and a solid circular bronze shaft (G_{br} = 40 GPa) are securely connected by a coupling at C as shown in Figure P5.67. A torque of T = 10 kN·m is applied to the rigid wheel B. If the coupling plates cannot rotate relative to one another, determine the angle of rotation of wheel B due to the applied torque.

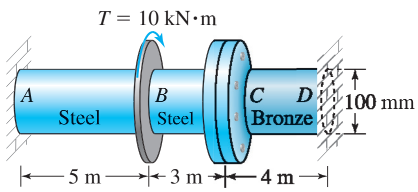


Figure P5.67

5.68 Two solid circular steel shafts (G_{st} = 80 GPa) and a solid circular bronze shaft (G_{br} = 40 GPa) are connected by a coupling at C as shown in Figure P5.67. A torque of T = 10 kN·m is applied to the rigid wheel B. If the coupling plates can rotate relative to one another by 0.5° before engaging, then what will be the angle of rotation of wheel B?

5.69 A solid steel shaft (G = 80 GPa) is securely fastened to a solid bronze shaft (G = 40 GPa) that is 2 m long, as shown in Figure P5.69. If T_{ext} = 10 kN · m, determine (a) the magnitude of maximum torsional shear stress in the shaft; (b) the rotation of the section at 1 m from the left wall.

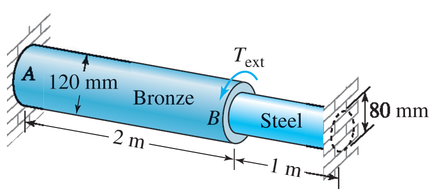


Figure P5.69

5.70 A solid steel shaft (G = 80 GPa) is securely fastened to a solid bronze shaft (G = 40 GPa) that is 2 m long, as shown in Figure P5.69. If the section at B rotates by 0.05 rad, determine (a) the maximum torsional shear strain in the shaft; (b) the applied torque T_{ext}.

5.71 Two shafts with shear moduli $G_1 = G$ and $G_2 = 2G$ are securely fastened at section B, as shown in Figure P5.71. In terms of T_{ext}, L, G, and d, find the magnitude of maximum torsional shear stress in the shaft and the rotation of the section at B.

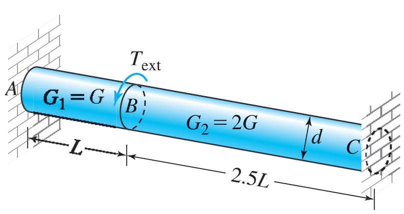


Figure P5.71

5.72 A uniformly distributed torque of q in.·lb/in. is applied to the entire shaft, as shown in Figure P5.72. In addition to the distributed torque a concentrated torque of $T = 3qL$ in.·lb is applied at section B. Let the shear modulus be G and the radius of the shaft r. In terms of q, L, G, and r determine (a) the rotation of the section at B; (b) the magnitude of maximum torsional shear stress in the shaft.

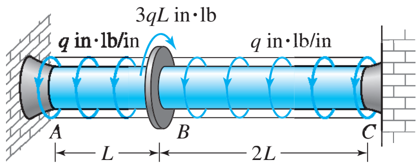


Figure P5.72

Design problems

5.73 A steel shaft (G_{st} = 80 GPa) and a bronze shaft (G_{br} = 40 GPa) are securely connected at B, as shown in Figure P5.69. The magnitude of maximum torsional shear stresses in steel and bronze are to be limited to 160 MPa and 60 MPa, respectively. Determine the maximum allowable torque T_{ext} to the nearest kN·m that can act on the shaft.

5.74 A steel shaft (G_{st} = 80 GPa) and a bronze shaft (G_{br} = 40 GPa) are securely connected at B, as shown in Figure P5.74. The magnitude of maximum torsional shear stresses in steel and bronze are to be limited to 160 MPa and 60 MPa, respectively, and the rotation of section B is limited to 0.05 rad. (a) Determine the maximum allowable torque T to the nearest kN·m that can act on the shaft if the diameter of the shaft is d = 100 mm. (b) What are the magnitude of maximum torsional shear stress and the maximum rotation in the shaft corresponding to the answer in part (a)?

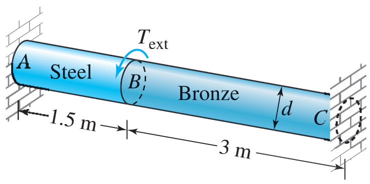


Figure P5.74

5.75 A steel shaft (G_{st} = 80 GPa) and a bronze shaft (G_{br} = 40 GPa) are securely connected at B, as shown in Figure P5.74. The magnitude of maximum torsional shear stresses in steel and bronze are to be limited to 160 MPa and 60 MPa, respectively, and the rotation of section B is limited to 0.05 rad. (a) Determine the minimum diameter d of the shaft to the nearest millimeter if the applied torque T = 20 kN · m. (b) What are the magnitude of maximum torsional shear stress and the maximum rotation in the shaft corresponding to the answer in part (a)?

5.76 The solid steel shaft shown in Figure P5.76 has a shear modulus of elasticity G = 80 GPa and an allowable torsional shear stress of 60 MPa. The allowable rotation of any section is 0.03 rad. The applied torques on the shaft are T_1 = 10 kN·m and T_2 = 25 kN·m. Determine (a) the minimum diameter d of the shaft to the nearest millimeter; (b) the magnitude of maximum torsional shear stress in the shaft and the maximum rotation of any section.

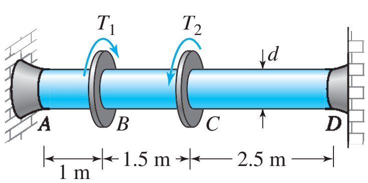


Figure P5.76

5.77 The diameter of the shaft shown in Figure P5.76 d = 80 mm. Determine the maximum values of the torques T_1 and T_2 to the nearest kN·m that can be applied to the shaft.

Composite Shafts

5.78 An aluminum tube and a copper tube, each having a thickness of 5 mm, are securely fastened to two rigid bars, as shown in Figure P5.78. The bars force the tubes to rotate by equal angles. The two tubes are 1.5 m long, and the mean diameters of the aluminum and copper tubes are 125 mm and 50 mm, respectively. The shear moduli for aluminum and copper are G_{al} = 28 GPa and G_{cu} = 40 GPa. Under the action of the applied couple section *B* of the two tubes rotates by an angle of 0.03 rad Determine (a) the magnitude of maximum torsional shear stress in aluminum and copper; (b) the magnitude of the couple that produced the given rotation.

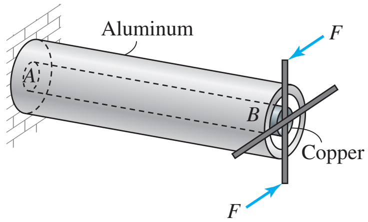


Figure P5.78

5.79 Solve Problem 5.78 using Equations (5.18a) and (5.18b).

5.80 An aluminum tube and a copper tube, each having a thickness of 5 mm, are securely fastened to two rigid bars, as shown in Figure P5.78. The bars force the tubes to rotate by equal angles. The two tubes are 1.5 m long and the mean diameters of the aluminum and copper tubes are 125 mm and 50 mm, respectively. The shear moduli for aluminum and copper are $G_{al} = 28$ GPa and $G_{cu} = 40$ GPa. The applied couple on the tubes shown in Figure P5.78 is $10\ \text{kN}\cdot\text{m}$. Determine (a) the magnitude of maximum torsional shear stress in aluminum and copper; (b) the rotation of the section at *B*.

5.81 Solve Problem 5.80 using Equations (5.18a) and (5.18b).

5.82 Solve Example 5.14 using Equations (5.18a) and (5.18b).

5.83 The composite shaft shown in Figure P5.83 is constructed from aluminum ($G_{al} = 4000$ ksi), bronze ($G_{br} = 6000$ ksi), and steel ($G_{st} = 12{,}000$ ksi). (a) Determine the rotation of the free end with respect to the wall. (b) Plot the torsional shear strain and the shear stress across the cross section

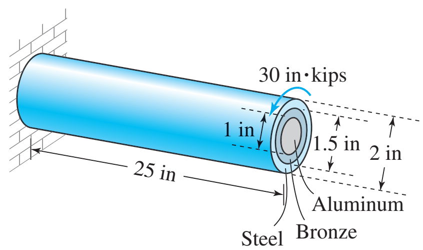


Figure P5.83

5.84 Solve Problem 5.83 using Equations (5.18a) and (5.18b).

5.85 If $T = 1500\ \text{N}\cdot\text{m}$ in Figure P5.85, determine (a) the magnitude of maximum torsional shear stress in cast iron and copper; (b) the rotation of the section at *D* with respect to the section at *A*.

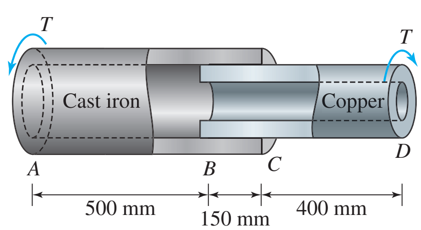


Figure P5.85

Shafts on multiple axis

5.86 Two steel (G = 80 GPa) shafts *AB* and *CD* of diameters 40 mm are connected with gears as shown in Figure P5.86. The radii of gears at *B* and *C* are 250 mm and 200 mm, respectively. The bearings at *E* and *F* offer no torsional resistance to the shafts. If an input torque of T_{ext} = 1.5 kN.m is applied at *D*, determine (a) the maximum torsional shear stress in *AB*; (b) the rotation of section at *D* with respect to the fixed section at *A*.

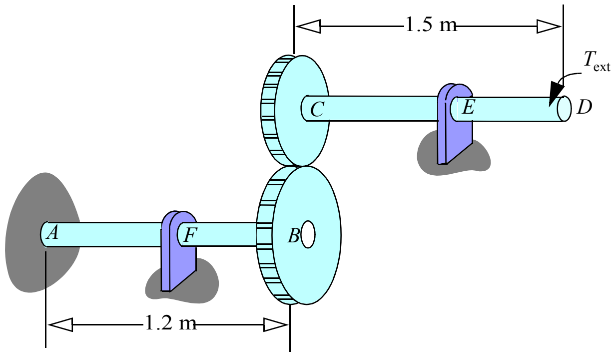


Figure P5.86

5.87 Two steel (G = 80 GPa) shafts *AB* and *CD* of diameters 40 mm are connected with gears as shown in Figure P5.86. The radii of gears at *B* and *C* are 250 mm and 200 mm, respectively. The bearings at *E* and *F* offer no torsional resistance to the shafts. The allowable shear stress in the shafts is 120 MPa. Determine the maximum torque *T* that can be applied at section *D*.

5.88 Two steel (G = 80 GPa) shafts *AB* and *CDE* of 1.5 in. diameters are connected with gears as shown in Figure P5.88. The radii of gears at *B* and *D* are 9 in. and 5 in., respectively. The bearings at *F, G,* and *H* offer no torsional resistance to the shafts. If an input torque of T_{ext} = 800 ft.lb is applied at D, determine (a) the maximum torsional shear stress in *AB*; (b) the rotation of section at *E* with respect to the fixed section at *C*.

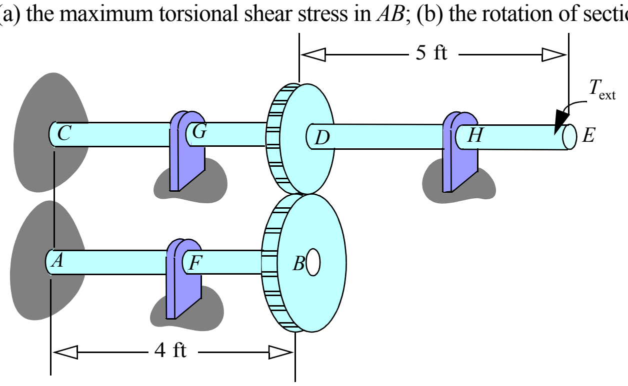


Figure P5.88

5.89 Two steel (G = 80 GPa) shafts *AB* and *CD* of 60 mm diameters are connected with gears as shown in Figure P5.89. The radii of gears at *B* and *D* are 175 mm and 125 mm, respectively. The bearings at *E* and *F* offer no torsional resistance to the shafts. If an input torque of T_{ext} = 2 kN.m is applied, determine (a) the maximum torsional shear stress in *AB*; (b) the rotation of section at *D* with respect to the fixed section at *C*.

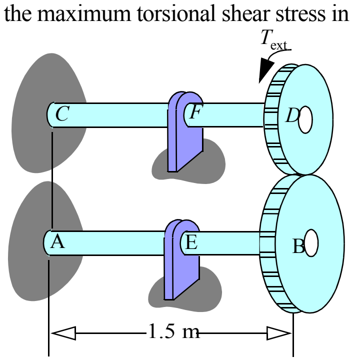


Figure P5.89

5.90 Two steel (G = 80 GPa) shafts *AB* and *CD* of 60 mm diameters are connected with gears as shown in Figure P5.89. The radii of gears at *B* and *D* are 175 mm and 125 mm, respectively. The bearings at *E* and *F* offer no torsional resistance to the shafts. The allowable shear stress in the shafts is 120 MPa. What is the maximum torque *T* that can be applied?

5.91 Two steel (G = 80 GPa) shafts *AB* and *CD* of equal diameters *d* are connected with gears as shown in Figure P5.89. The radii of gears at *B* and *D* are 175 mm and 125 mm, respectively. The bearings at *E* and *F* offer no torsional resistance to the shafts. The allowable shear stress in the shafts is 120 MPa and the input torque is *T* = 2 kN.m. Determine the minimum diameter *d* to the nearest millimeter.

Stress concentration

5.92 The allowable shear stress in the stepped shaft shown Figure P5.92 is 17 ksi. Determine the smallest fillet radius that can be used at section B. Use the stress concentration graphs given in Section C.4.3.

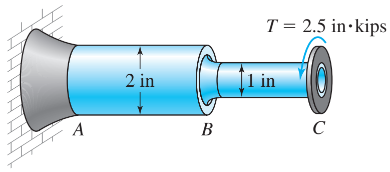


Figure P5.92

5.93 The fillet radius in the stepped shaft shown in Figure P5.93 is 6 mm. Determine the maximum torque that can act on the rigid wheel if the allowable shear stress is 80 MPa and the modulus of rigidity is 28 GPa. Use the stress concentration graphs given in Section C.4.3.

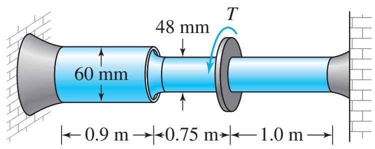


Figure P5.93

5.4* TORSION OF THIN-WALLED TUBES

The sheet metal skin on a fuselage, the wing of an aircraft, and the shell of a tall building are examples in which a body can be analyzed as a thin-walled tube. By thin wall we imply that the thickness t of the wall is smaller by a factor of at least 10 in comparison to the length b of the biggest line that can be drawn across two points on the cross section, as shown in Figure 5.50*a*. By a *tube* we imply that the length L is at least 10 times that of the cross-sectional dimension b.
We assume that this thin-walled tube is subjected to only torsional moments.

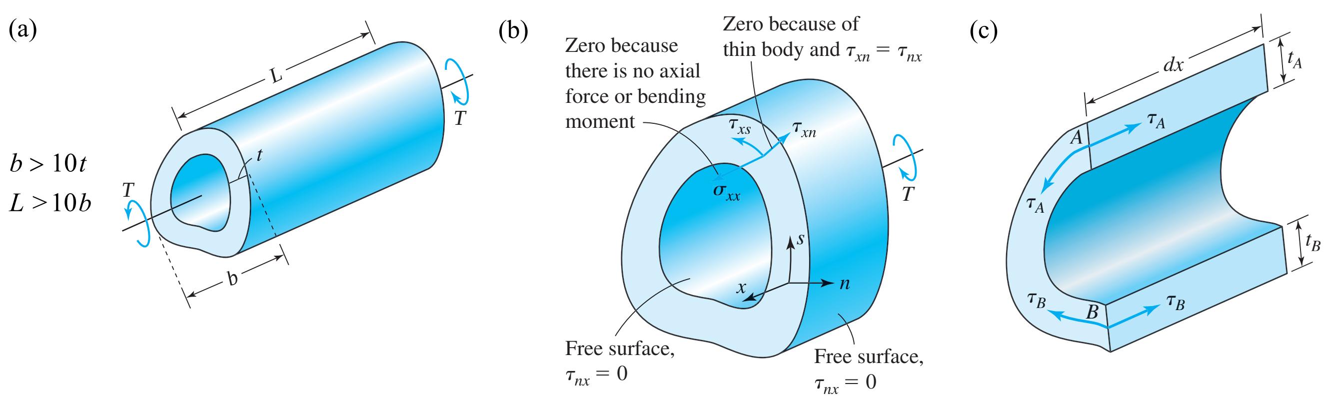


Figure 5.50 (a) Torsion of thin-walled tubes. (b) Deducing stress behavior in thin-walled tubes. (c) Deducing constant shear flow in thin-walled tubes.

The walls of the tube are bounded by two free surfaces, and hence by the symmetry of shear stresses the shear stress in the normal direction τ_{xn} must go to zero on these bounding surfaces, as shown in Figure 5.50*b*.This does not imply that τ_{xn} is zero in the interior. However, because the walls are thin, we approximate τ_{xn} as zero everywhere. The normal stress σ_{xx} would be equivalent to an internal axial force or an internal bending moment. Since there is no external axial force or bending moment, we approximate the value of σ_{xx} as zero.

Figure 5.50*b* shows that the only nonzero stress component is τ_{xs}. It can be assumed uniform in the n direction because the tube is thin. Figure 5.50*c* shows a free-body diagram of a differential element with an imaginary cut through points A and B. By equilibrium of forces in the x direction we obtain

$$\tau_A(t_A\, dx) = \tau_B(t_B\, dx) \tag{5.24a}$$

$$\tau_A t_A = \tau_B t_B \tag{5.24b}$$

$$q_A = q_B \tag{5.24c}$$

The quantity $q = \tau_{xs} t$ is called shear flow[5] and has units of force per unit length. Equation (5.24c) shows that shear flow is uniform at a given cross section.

We can replace the shear stresses (shear flow) by an equivalent internal torque, as shown in Figure 5.51. The line OC is perpendicular to the line of action of the force $d\boldsymbol{V}$, which is in the tangent direction to the arc at that point. Noting that the shear flow is a constant, we take it outside the integral sign,

$$\boldsymbol{T} = \oint dT = \oint q(h\ ds) = q\oint 2\ dA_E = 2qA_E \qquad \text{or} \qquad q = \frac{\boldsymbol{T}}{2A_E} \tag{5.25}$$

We thus obtain

$$\boxed{\tau_{xs} = \frac{\boldsymbol{T}}{2tA_E}} \tag{5.26}$$

where $\boldsymbol{T}$ is the internal torque at the section containing the point at which the shear stress is to be calculated, A_E is the area enclosed by the centerline of the tube, and t is the thickness at the point where the shear stress is to be calculated.

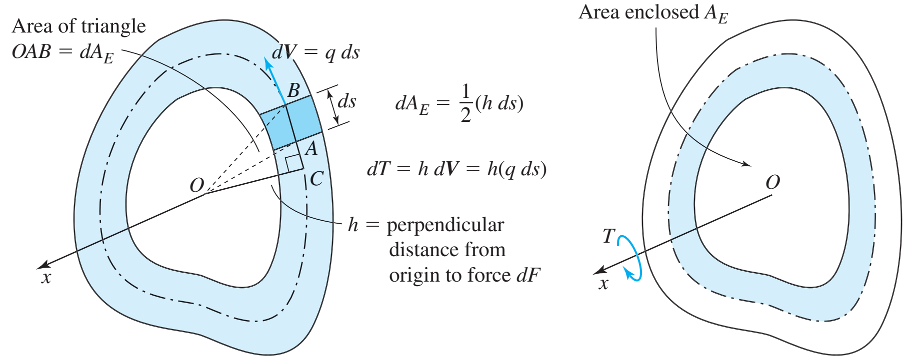


Figure 5.51 Equivalency of internal torque and shear stress (flow).

The thickness t can vary with different points on the cross section provided the assumption of thin-walled is not violated. If the thickness varies, then the shear stress will not be constant on the cross section, even though the shear flow is constant.

EXAMPLE 5.15

A semicircular thin tube is subjected to torques as shown in Figure 5.52. Determine: (a) The maximum torsional shear stress in the tube. (b) The torsional shear stress at point O. Show the results on a stress cube.

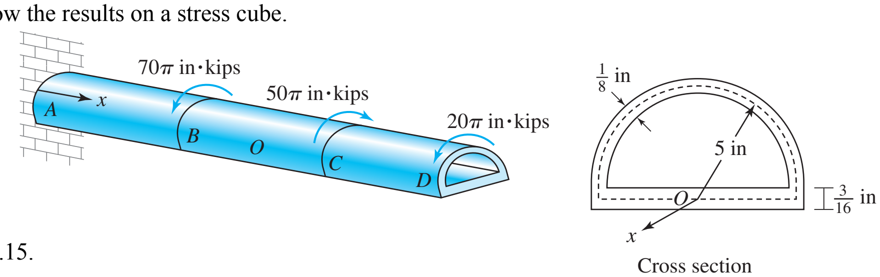


Figure 5.52 Thin-walled tube in Example 5.15.

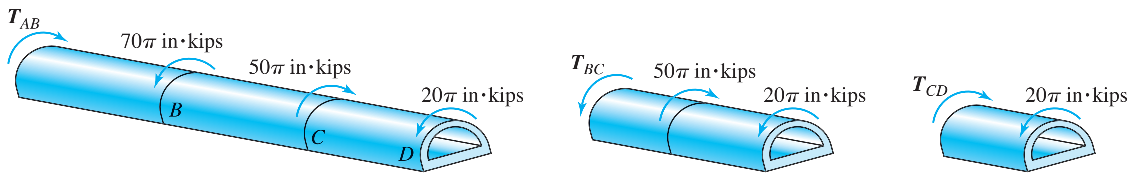
PLAN

From Equation 5.26 we know that the maximum torsional shear stress will exist in a section where the internal torque is maximum and the thickness minimum. To determine the maximum internal torque, we make cuts in AB, BC, and CD and draw free-body diagrams by taking the right part of each cut to avoid calculating the wall reaction.

SOLUTION

[5]This terminology is from fluid mechanics, where an incompressible ideal fluid has a constant flow rate in a channel.

Figure 5.53 shows the free-body diagrams after making an imaginary cut and taking the right part.



$T_{AB} + 50\pi - 70\pi - 20\pi = 0$
$T_{AB} = 40\pi$ in·kips

$-T_{BC} + 50\pi - 20\pi = 0$
$T_{BC} = 30\pi$ in·kips

$T_{CD} - 20\pi = 0$
$T_{CD} = 20\pi$ in·kips

Figure 5.53 Internal torque calculations in Example 5.15.

(a) The maximum torque is in *AB* and the minimum thickness is $\frac{1}{8}$ in. The enclosed area is $A_E = \pi(5\text{ in.})^2/2 = 12.5\pi\text{ in.}^2$. From Equation 5.26 we obtain

$$\tau_{max} = \frac{(40\pi\text{ in.}\cdot\text{kips})}{(12.5\pi\text{ in.}^2)(\frac{1}{8}\text{ in.})} \tag{E1}$$

ANS. $\tau_{max} = 25.6$ ksi

(b) At point *O* the internal torque is T_{BC} and $t = \frac{3}{16}$ in. We obtain the shear stress at *O* as

$$\tau_O = \frac{(30\pi\text{ in.}\cdot\text{kips})}{(12.5\pi\text{ in.}^2)(\frac{3}{16}\text{ in.})} \tag{E2}$$

ANS. $\tau_O = 12.8\ ksi$

Figure 5.54 shows part of the tube between sections *B* and *C*. Segment *BO* would rotate counterclockwise with respect to segment *OC*. The shear stress must be opposite to this possible motion and hence in the clockwise direction, as shown. The direction on the other surfaces can be drawn using the observation that the symmetric pair of shear stress components either point toward the corner or away from it.

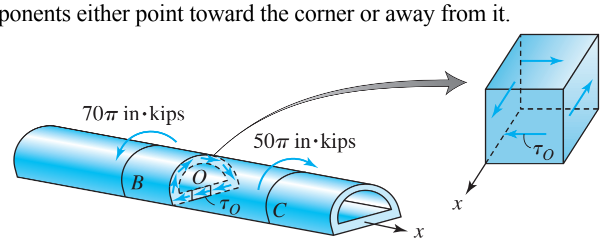


Figure 5.54 Direction of shear stress in Example 5.15.

COMMENT

1. The shear flow in the cross-section containing point *O* is a constant over the entire cross-section. The magnitude of torsional shear stress at point *O* however will be two-thirds that of the value of the shear stress in the circular part of the cross-section because of the variation in wall thickness.

PROBLEM SET 5.4

Torsion of thin-walled tubes

5.94 Calculate the magnitude of the maximum torsional shear stress if the cross section shown in Figure P5.94 is subjected to a torque $T = 100$ in.·kips.

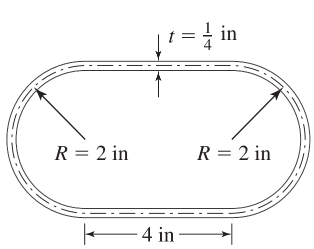


Figure P5.94

5.95 Calculate the magnitude of the maximum torsional shear stress if the cross section shown in Figure P5.95 is subjected to a torque $T = 900\ \text{N}\cdot\text{m}$.

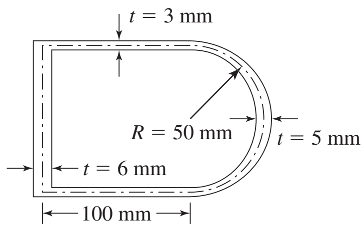


Figure P5.95

5.96 Calculate the magnitude of the maximum torsional shear stress if the cross section shown in Figure P5.96 is subjected to a torque $T = 15\ \text{kN}\cdot\text{m}$.

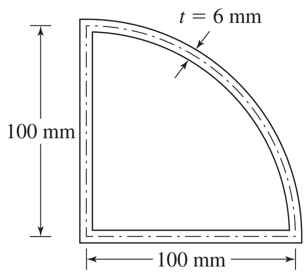


Figure P5.96

5.97 A tube of uniform thickness t and cross section shown in Figure P5.97 has a torque T applied to it. Determine the maximum torsional shear stress in terms of t, a, and T.

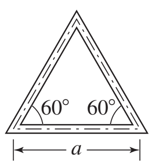


Figure P5.97

5.98 A tube of uniform thickness t and cross section shown in Figure P5.98 has a torque T applied to it. Determine the maximum torsional shear stress in terms of t, a, and T.

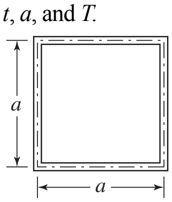


Figure P5.98

5.99 A tube of uniform thickness t and cross section shown in Figure P5.99 has a torque T applied to it. Determine the maximum torsional shear stress in terms of t, a, and T.

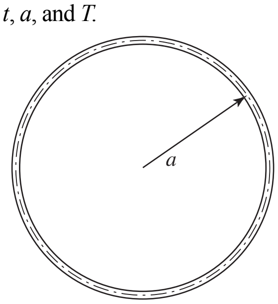


Figure P5.99

5.100 The tube of uniform thickness t shown in Figure P5.100 has a torque T applied to it. Determine the maximum torsional shear stress in terms of t, a, b, and T.

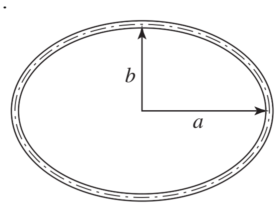


Figure P5.100

5.101 A hexagonal tube of uniform thickness is loaded as shown in Figure P5.101. Determine the magnitude of the maximum torsional shear stress in the tube

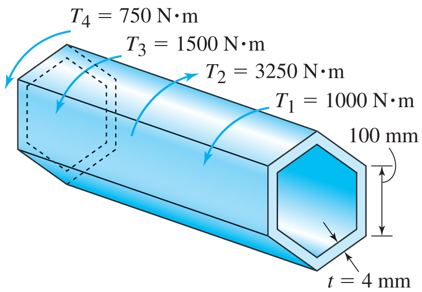


Figure P5.101

5.102 A rectangular tube is loaded as shown in Figure P5.102. Determine the magnitude of the maximum torsional shear stress.

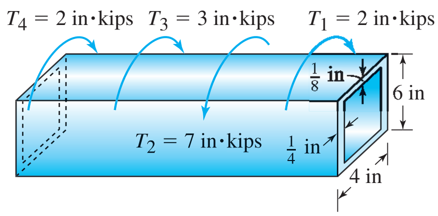


Figure P5.102

5.103 The three tubes shown in Problems 5.97 through 5.99 are to be compared for the maximum torque-carrying capability, assuming that all tubes have the same thickness t, the maximum torsional shear stress in each tube can be τ, and the amount of material used in the cross section of each tube is A. (a) Which shape would you use? (b) What is the percentage torque carried by the remaining two shapes in terms of the most efficient structural shape?

5.5* CONCEPT CONNECTOR

Like so much of science, the theory of torsion in shafts has a history filled with twists and turns. Sometimes experiments led the way; sometimes logic pointed to a solution. As so often, too, serendipity guided developments. The formulas were developed empirically, to meet a need—but not in the mechanics of materials. Instead, a scientist had a problem to solve in electricity and magnetism, and torsion helped him measure the forces. It was followed with an analytical development of the theory for circular and non-circular shaft cross sections that stretched over a hundred years. The description of the history concludes with an experimental technique used in the calculation of torsional rigidity, even for shafts of arbitrary shapes.

5.5.1 History: Torsion of Shafts

It seems fitting that developments begin with Charles-Augustin Coulomb (Figure 5.55). Coulomb, who first differentiated shear stress from normal stress (see Section 1.6.1), also studied torsion, in which shear stress is the dominant component. In 1781 Coulomb started his research in electricity and magnetism. To measure the small forces involved, he devised a very sensitive *torsion balance*. A weight was suspended by a wire, and a pointer attached to the weight indicated the wire's angular rotation.

Figure 5.55 Charles-Augustin Coulomb.

The design of this torsion balance led Coulomb to investigate the resistance of a wire in torsion. He assumed that the resistance torque (or internal torque ***T***) in a twisted wire is proportional to the angle of twist ϕ. To measure the changes, he twisted the wire by a small angle and set it free to oscillate, like a pendulum. After validating his formula experimentally, thus confirming his assumption, he proceeded to conduct a parametric study with regard to the length L and the diameter D of the wire and developed the following formula $\boldsymbol{T} = (\mu D^4/L)\phi$, where μ is a material constant. If we substitute $d\phi/dx = \phi/L$ and $J = \pi D^4/32$ into Equation 5.9 and compare our result with Coulomb's formula, we see that Coulomb's material constant is $\mu = \pi G/32$.

Coulomb's formula, although correct, was so far only an empirical relationship. The analytical development of the theory for circular shafts is credited to Alphonse Duleau. Duleau, born in Paris the year of the French revolution, was commissioned in 1811 to design a forged iron bridge over the Dordogne river, in the French city of Cubzac. Duleau had graduated from the École Polytechnique, one of the early engineering schools. Founded in 1794, it had many pioneers in the mechanics of materials among its faculty and students. At the time, there was little or no data on the behavior of bars under the loading conditions needed for bridge design. Duleau therefore conducted extensive experiments on tension, compression, flexure, torsion, and elastic stability. He also compared bars of circular, triangular, elliptical, and rectangular cross section. In 1820 he published his results.

In this paper Duleau developed Coulomb's torsion formula analytically, starting with our own Assumptions 1 and 3 (see page 215), that is, cross sections remain planes and radial lines remain straight during small twists to circular bars. He also established that these assumptions are not valid for noncircular shafts.

Augustin-Louis Cauchy, whose contributions to the mechanics of materials we have encountered in several chapters, was also interested in the torsion of rectangular bars. Cauchy showed that the cross section of a rectangular bar does not remain a plane. Rather, it warps owing to torsional loads.

Jean Claude Saint-Venant proposed in 1855 the displacement behavior we encountered in Problem 5.53. Building on the observations of Coulomb, Duleau, and Cauchy, he developed torsion formulas for a variety of shapes. Saint-Venant's assumed a displacement function that incorporates some features based on experience and empirical information but containing sufficient unknown parameters to satisfy equations of elasticity, an approach now called *Saint-Venant's semi-inverse method.*

Ludwig Prandtl (1875-1953) is best known for his work in aerodynamics, but the German physicist's interests ranged widely in engineering design. He originated boundary-layer theory in fluid mechanics. He also invented the wind tunnel and its use in airplane design. In 1903 Prandtl was studying the differential equations that describe the equilibrium of a soap film, a thin-walled membrane. He found that these are similar to torsion equations derived using Saint-Venant's semi-inverse method. Today, Prandtl's *membrane analogy* is used to obtain torsional rigidities for complex cross sections simply from experiments on soap films. Handbooks list torsional rigidities for variety of shapes, many of which were obtained from membrane analogy.

We once more see that theory is the outcome of a serendipitous combination of experimental and analytical thinking.

5.6 CHAPTER CONNECTOR

In this chapter we established formulas for torsional deformation and stresses in circular shafts. We saw that the calculation of stresses and relative deformation requires the calculation of the internal torque at a section. For statically determinate shafts, the internal torque can be calculated in either of two ways. In the first, we make an imaginary cut and draw an appropriate free-body diagram. In the second, we draw a torque diagram. In statically indeterminate single-axis shafts, the internal torque expression contains an unknown reaction torque, which has to be determined using the compatibility equation. For single-axis shafts, the relative rotation of a section at the right wall with respect to the rotation at the left wall is zero. This result is the compatibility equation.

We also saw that torsional shear stress should be drawn on a stress element. This approach will be important in studying stress and strain transformation in future chapters. In Chapter 8, on *stress transformation*, we will first find torsional shear stress using the stress formula from this chapter. We then find stresses on inclined planes, including planes with maximum normal stress. In Chapter 9, on *strain transformation*, we will find the torsional shear strain and then consider strains in different coordinate systems, including coordinate systems in which the normal strain is maximum. In Section 10.1, we will consider the combined loading problems of axial, torsion, and bending. This will lead to the design of simple structures that may be either determinate or indeterminate.

POINTS AND FORMULAS TO REMEMBER

- Our theory describing the torsion of shafts is limited to: (1) slender shafts of circular cross sections; and (2) regions away from the neighborhood of stress concentration. The variation in cross sections and external torques is gradual.

$$\boldsymbol{T} = \int_A \rho \tau_{x\theta}\, dA \quad (5.1) \qquad \phi = \phi(x) \quad (5.2) \qquad \text{small strain } \gamma_{x\theta} = \rho \frac{d\phi}{dx} \quad (5.3)$$

- where $\boldsymbol{T}$ is the internal torque that is positive counterclockwise with respect to the outward normal to the imaginary cut surface, ϕ is the angle of rotation of the cross section that is positive counterclockwise with respect to the x axis, $\tau_{x\theta}$ and $\gamma_{x\theta}$ are the torsional shear stress and strain in polar coordinates, and ρ is the radial coordinate of the point where shear stress and shear strain are defined.
- Equations (5.1), (5.2), and (5.3) are independent of material model.
- Torsional shear strain varies linearly with radial coordinate across the cross section.
- Torsional shear strain is maximum at the outer surface of the shaft.
- The formulas below are valid for shafts with material that is linear, elastic, and isotropic and has a homogeneous cross section:

$$\frac{d\phi}{dx} = \frac{\boldsymbol{T}}{GJ} \quad (5.9) \qquad \tau_{x\theta} = \frac{\boldsymbol{T}\rho}{J} \quad (5.10) \qquad \phi_2 - \phi_1 = \frac{\boldsymbol{T}(x_2 - x_1)}{GJ} \quad (5.12)$$

- where G is the shear modulus of elasticity, and J is the polar moment of the cross section given by $J = (\pi/2)(R_o^4 - R_i^4)$, R_o and R_i being the outer and inner radii of a hollow shaft.
- The quantity GJ is called torsional rigidity.
- If T, G, or J *change* with x, we find the relative rotation of a cross section by integration of Equation (5.9).
- If T, G, and J *do not change* between x_1 and x_2, we use Equation (5.12) to find the relative rotation of a cross section.
- Torsional shear stress varies linearly with radial coordinate across the homogeneous cross section, reaching a maximum value on the outer surface of the shaft.