CHAPTER THREE

MECHANICAL PROPERTIES OF MATERIALS

Learning objectives

1. Understand the qualitative and quantitative descriptions of mechanical properties of materials.
2. Learn the logic of relating deformation to external forces.

The ordinary wire and rubber stretch cord in Figure 3.1 have the same undeformed length and are subjected to the same loads. Yet the rubber deforms significantly more, which is why we use rubber cords to tie luggage on top of a car. As the example shows, before we can relate deformation to applied forces, we must first describe the mechanical properties of materials.

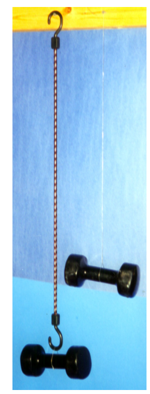


Figure 3.1 Material impact on deformation.

In engineering, adjectives such as *elastic*, *ductile*, or *tough* have very specific meanings. These terms will be our *qualitative* description of materials. Our *quantitative* descriptions will be the equations relating stresses and strains. Together, these description form the material model (Figure 3.2). The parameters in the material models are determined by the least-square method (see Appendix B.3) to fit the best curve through experimental observations. In this chapter, we develop a simple model and learn the logic relating deformation to forces. In later chapters, we shall apply this logic to axial members, shafts, and beams and obtain formulas for stresses and deformations.

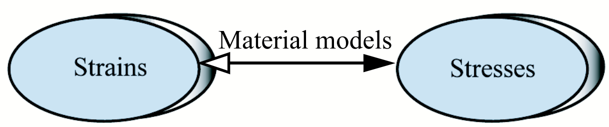


Figure 3.2 Relationship of stresses and strains.

3.1 MATERIALS CHARACTERIZATION

The American Society for Testing and Materials (ASTM) specifies test procedures for determining the various properties of a material. These descriptions are guidelines used by experimentalists to obtain reproducible results for material properties. In this section, we study the tension and compression tests, which allow us to determine many parameters relating stresses and strains.

3.1.1 Tension Test

In the **tension test**, standard specimen are placed in a tension-test machine, where they are gripped at each end and pulled in the axial direction. Figure 3.3 shows two types of standard geometry: a specimen with a rectangular cross-section and specimen with a circular cross-section.

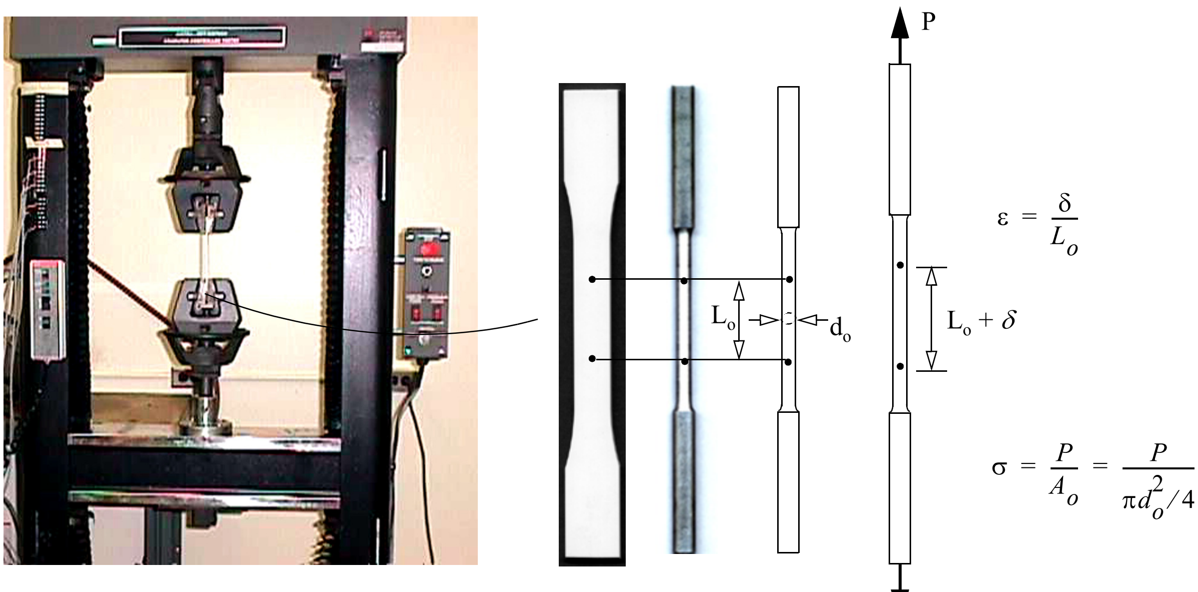


Figure 3.3 Tension test machine and specimen. (Courtesy Professor I. Miskioglu.)

Two marks are made in the central region, separated by the **gage length** L_0. The deformation δ is movement of the two marks. For metals, such as aluminum or steel, ASTM recommends a gage length $L_0 = 2$ in. and diameter $d_0 = 0.5$ in. The normal strain ε is the deformation δ divided by L_0.

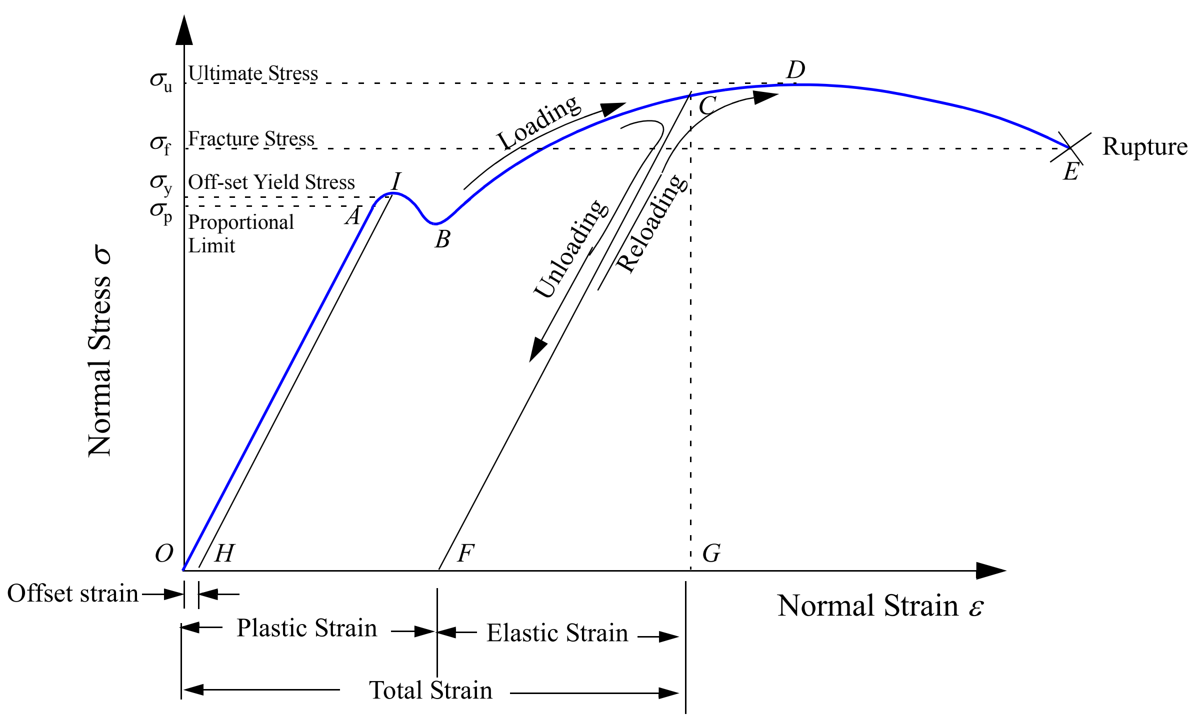


Figure 3.4 Stress–strain curve.

The tightness of the grip, the symmetry of the grip, friction, and other local effects are assumed and are observed to die out rapidly with the increase in distance from the ends. This dissipation of local effects is further facilitated by the gradual change in the cross-section. The specimen is designed so that its central region is in a uniform state of axial stress. The normal stress is calculated by dividing the applied force P by the area of cross section A_0, which can be found from the specimen's width or diameter.

The tension test may be conducted by controlling the force P and measuring the corresponding deformation δ. Alternatively, we may control the deformation δ by the movement of the grips and measuring the corresponding force P. The values of force P and deformation δ are recorded, from which normal stress σ and normal strain ε are calculated. Figure 3.4 shows a typical stress–strain (σ-ε) curve for metal.

As the force is applied, initially a straight line (OA) is obtained. The end of this linear region is called the **proportional limit**. For some metals, the stress may then decrease slightly (the region *AB*), before increasing once again. The largest stress (point *D* on the curve) is called the **ultimate stress**. In a force-controlled experiment, the specimen will suddenly break at the ultimate stress. In a displacement-controlled experiment we will see a decrease in stress (region *DE*). The stress at breaking point *E* is called **fracture** or **rupture stress**.

Elastic and plastic regions

If we load the specimen up to any point along line *OA*—or even a bit beyond—and then start unloading, we find that we retrace the stress–strain curve and return to point *O*. In this **elastic region**, the material regains its original shape when the applied force is removed.

If we start unloading only after reaching point *C*, however, then we will come down the straight line *FC*, which will be parallel to line *OA*. At point *F*, the *stress* is zero, but the *strain* is nonzero. *C* thus lies in the **plastic region** of the stress-strain curve, in which the material is deformed permanently, and the permanent strain at point *F* is the **plastic strain**. The region in which the material deforms permanently is called **plastic region**. The total strain at point C is sum the plastic strain (*OF*) and an additional **elastic strain** (*FG*)

The point demarcating the elastic from the plastic region is called the **yield point**. The stress at yield point is called the **yield stress**. In practice, the yield point may lie anywhere in the region *AB*, although for most metals it is close to the proportional limit. For many materials it may not even be clearly defined. For such materials, we mark a prescribed value of offset strain recommended by ASTM to get point *H* in Figure 3.4. Starting from *H* we draw a line (*HI*) parallel to the linear part (*OA*) of the stress–strain curve. **Offset yield stress** would correspond to a plastic strain at point *I*. Usually the offset strain is given as a percentage. A strain of 0.2% equals $\varepsilon = 0.002$ (as described in Chapter 2).

It should be emphasized that *elastic* and *linear* are two distinct material descriptions. Figure 3.5*a* shows the stress–strain curve for a soft rubber that can stretch several times its original length and still return to its original geometry. Soft rubber is thus *elastic* but *nonlinear* material.

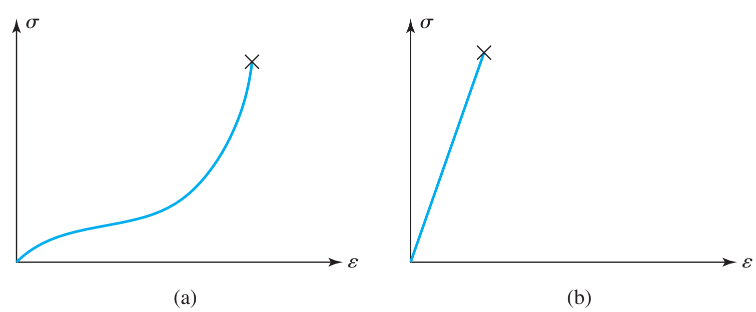


Figure 3.5 Examples of nonlinear and brittle materials. (*a*) Soft rubber. (*b*) Glass.

Ductile and brittle materials

Ductile materials, such as aluminum and copper, can undergo large *plastic* deformations before fracture. (Soft rubber can undergo large deformations but it is not a ductile material.) Glass, on the other hand, is **brittle**: it exhibits little or no plastic deformation as shown in Figure 3.5*b*. A material's ductility is usually described as percent elongation before rupture. The elongation values of 17% for aluminum and 35% for copper before rupture reflect the large plastic strains these materials undergo before rupture, although they show small elastic deformation as well.

Recognizing ductile and brittle material is important in design, in order to characterize failure as we shall see in Chapters 8 and 10. A ductile material usually yields when the maximum shear stress exceeds the yield *shear stress*. A brittle material usually ruptures when the maximum *tensile normal stress* exceeds the ultimate tensile stress.

Hard and soft materials

A material **hardness** is its resistance to scratches and indentation (*not* its strength). In Rockwell test, the most common hardness test, a hard indenter of standard shape is pressed into the material using a specified load. The depth of indentation is measured and assigned a numerical scale for comparing hardness of different materials.

A soft material can be made harder by gradually increasing its yield point by **strain hardening**. As we have seen, at point *C* in Figure 3.4 the material has a permanent deformation even after unloading. If the material now is reloaded, point *C* becomes the new yield point, as additional plastic strain will be observed only after stress exceeds this point. Strain hardening is used, for example, to make aluminum pots and pans more durable. In the manufacturing process, known as *deep drawing,* the aluminum undergoes large plastic deformation. Of course, as the yield point increases, the remaining plastic deformation before fracture decreases, so the material becomes more brittle.

True stress and true strain

We noted that stress decreases with increasing strain between the ultimate stress and rupture (region *DE in* Figure 3.4). However, this decrease is seen only if we plot Cauchy's stress versus engineering strain. (Recall that Cauchy's stress is the load P divided by the original undeformed cross-sectional area.) An alternative is to plot true stress versus true strain, calculated using the actual, deformed cross-section and length (Section 1.6 and Problem 2.82). In such a plot, the stress in region DE continues to increase with increasing strain and just as in region *BD*.

Past ultimate stress a specimen also undergoes a sudden decease in cross-sectional area called **necking**. Figure 3.6 shows necking in a broken specimen from a tension test.

Figure 3.6 Specimen showing necking. (Courtesy Professor J. B. Ligon.)

3.1.2 Material Constants

Hooke's law give the relationship between normal stress and normal strain for the linear region:

$$\sigma = E\varepsilon \tag{3.1}$$

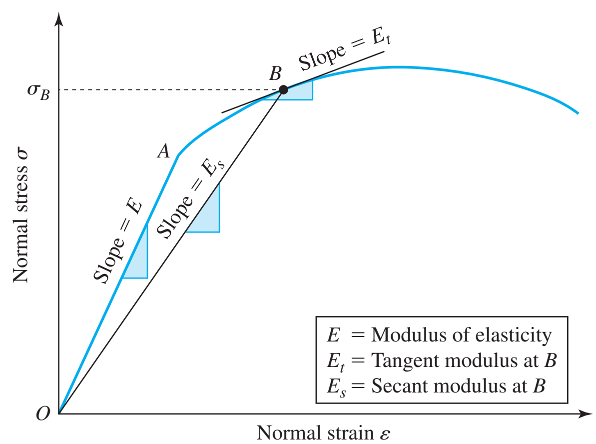


Figure 3.7 Different material moduli.

where *E* is called **modulus of elasticity** or **Young's modulus**. It represents the slope of the straight line in a stress–strain curve, as shown in Figure 3.7. Table 3.1 shows the moduli of elasticity of some typical engineering materials, with wood as a basis of comparison.

In the nonlinear regions, the stress–strain curve is approximated by a variety of equations as described in Section 3.11. The choice of approximation depends on the need of the analysis being performed. The two constants that are often used are shown in Figure 3.7. The slope of the tangent drawn to the stress–strain curve at a given stress value is called the **tangent modulus**. The slope of the line that joins the origin to the point on the stress–strain curve at a given stress value is called the **secant modulus.**

TABLE 3.1 **Comparison of moduli of elasticity for typical materials**

Material	Modulus of Elasticity (10^3 ksi)	Modulus Relative to Wood
Rubber	0.12	0.06
Nylon	0.60	0.30
Adhesives	1.10	0.55
Soil	1.00	0.50
Bones	1.86	0.93
Wood	2.00	1.00
Concrete	4.60	2.30
Granite	8.70	4.40
Glass	10.00	5.00
Aluminum	10.00	5.00
Steel	30.00	15.00

Figure 3.8 shows that the elongation of a cylindrical specimen in the longitudinal direction (direction of load) causes contraction in the lateral (perpendicular to load) direction and vice versa. The ratio of the two normal strains is a material constant called the **Poisson's ratio**, designated by the Greek letter ν (nu):

$$\nu = -\left(\frac{\varepsilon_{\text{lateral}}}{\varepsilon_{\text{longitudinal}}}\right) \tag{3.2}$$

Poisson's ratio is a dimensionless quantity that has a value between 0 and $\frac{1}{2}$ for most materials, although some composite materials can have negative values for ν. The theoretical range for Poisson's ration is $-1 \le \nu \le \frac{1}{2}$.

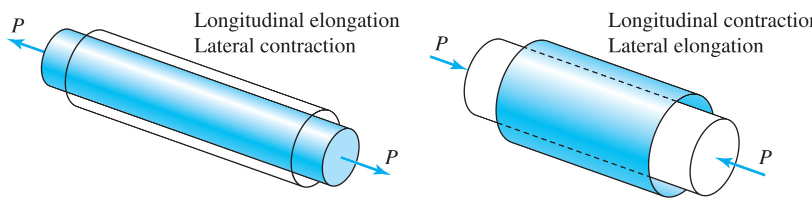


Figure 3.8 Poisson effect.

To establish the relationship between shear stress and shear strain, a torsion test is conducted using a machine of the type shown in Figure 3.9. On a plot of shear stress τ versus shear strain γ, we obtain a curve similar to that shown in Figure 3.4. In the linear region

$$\tau = G\gamma \tag{3.3}$$

where G is the **shear modulus of elasticity** or **modulus of rigidity**.

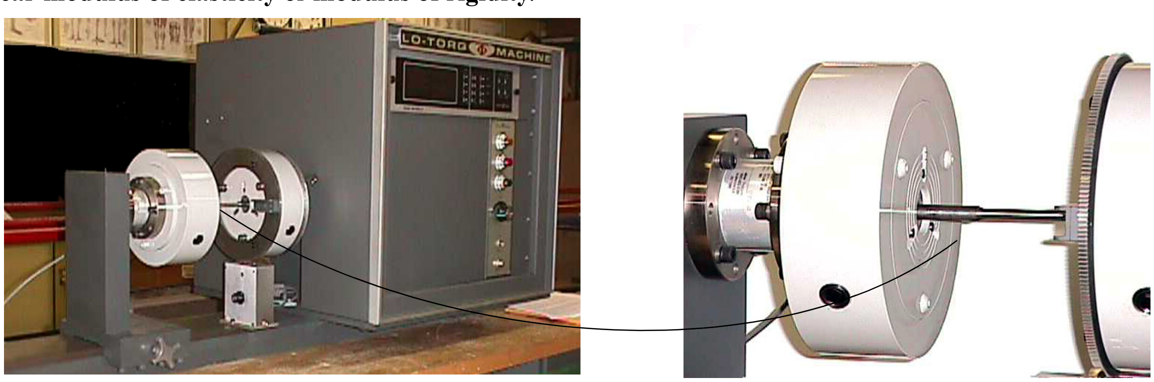


Figure 3.9 Torsion testing machine. (Courtesy Professor I. Miskioglu.)

3.1.3 Compression Test

We can greatly simplify analysis by assuming material behavior to be the same in tension and compression. This assumption of similar tension and compression properties works well for the values of material constants (such as E and ν). Hence the stress and deformation formulas developed in this book can be applied to members in tension and in compression. However, the compressive strength of many brittle materials can be very different from its tensile strength. In ductile materials as well the stress reversal from tension to compression in the plastic region can cause failure.

Figure 3.10*a* shows the stress–strain diagrams of two brittle materials. Notice the moduli of elasticity (the slopes of the lines) is the same in tension and compression. However, the compressive strength of cast iron is four times its tensile strength, while concrete can carry compressive stresses up to 5 ksi but has negligible tensile strength. Reinforcing concrete with steel bars can help, because the bars carry most of the tensile stresses.

Figure 3.10*b* shows the stress–strain diagrams for a ductile material such as mild steel. If compression test is conducted without unloading, then behavior under tension and compression is nearly identical: modulus of elasticity, yield stress, and ultimate stress are much the same. However, if material is loaded past the yield stress (point A), up to point B and then unloaded, the stress-strain diagram starts to curve after point C in the compressive region

Suppose we once more reverse loading direction, but starting at point D, which is at least $2\sigma_{\text{yield}}$ below point B, and ending at point F, where there is no applied load. The plastic strain is now less than that at point C. In fact, it is conceivable that the loading–unloading cycles can return the material to point O with no plastic strain. Does that mean we have the same material as the one we started with? No! The internal structure of the material has been altered significantly. Breaking of the material below the ultimate stress by load cycle reversal in the plastic region is called the **Bauschinger effect**. Design therefore usually precludes cyclic loading into the plastic region. Even in the elastic region, cyclic loading can cause failure due to fatigue (see Section 3.10).

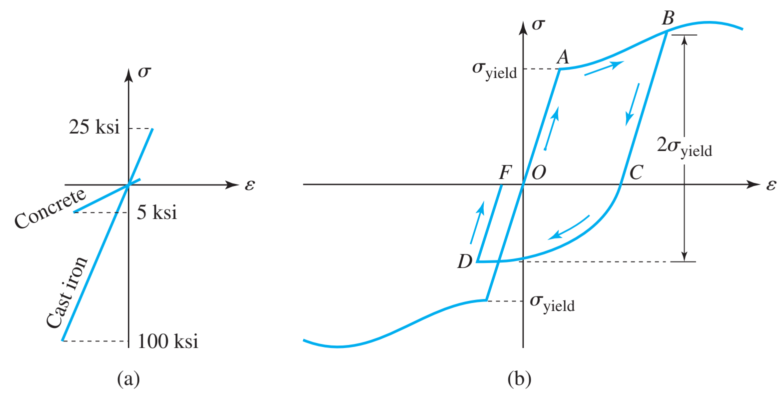


Figure 3.10 Differences in tension and compression. (*a*) Brittle material. (*b*) Ductile material.

EXAMPLE 3.1

A tension test was conducted on a circular specimen of titanium alloy. The gage length of the specimen was 2 in. and the diameter in the test region before loading was 0.5 in. Some of the data from the tension test are given in Table 3.2, where P is the applied load and δ is the corresponding deformation. Calculate the following quantities: (a) Stress at proportional limit. (b) Ultimate stress. (c) Yield stress at offset strain of 0.4%. (d) Modulus of elasticity. (e) Tangent and secant moduli of elasticity at a stress of 136 ksi. (f) Plastic strain at a stress of 136 ksi.

TABLE 3.2 Tension test data in Example 3.1

#	P (kips)	δ (10^{-3} in.)
1	0.0	0.0
2	5.0	3.2
3	15.0	9.5
4	20.0	12.7
5	24.0	15.3
6	24.5	15.6
7	25.0	15.9
8	25.2	16.9
9	25.4	19.7
10	26.0	28.5
11	26.5	36.9
12	27.0	46.5
13	27.5	58.3
14	28.0	75.2
15	28.2	87.1
16	28.3	100.0
17	28.2	112.9
18	28.0	124.8

TABLE 3.3 Stress and strain in Example 3.1

#	σ (ksi)	ε (10^{-3})
1	0.0	0.0
2	25.5	1.6
3	76.4	4.8
4	101.9	6.4
5	122.2	7.6
6	124.8	7.8
7	127.3	8.0
8	128.3	8.5
9	129.9	10.5
10	132.4	14.3
11	135.0	18.4
12	137.5	23.3
13	140.1	29.1
14	142.6	37.6
15	143.6	43.5
16	144.0	50.0
17	143.6	56.5
18	142.6	62.4

PLAN

We can divide the column of load P by the cross-sectional area to get the values of stress. We can divide the column of deformation δ by the gage length of 2 in. to get strain. We can plot the values to obtain the stress–strain curve and calculate the quantities, as described in Section 3.1.

SOLUTION

We divide the load column by the cross-sectional area $A = \pi(0.5 \text{ in.})^2/4 = 0.1964 \text{ in.}^2$ to obtain stress σ, and the deformation column by the gage length of 2 in. to obtain strain ε, as shown in Table 3.3, which is obtained using a spread sheet. Figure 3.11 shows the corresponding stress–strain curve.

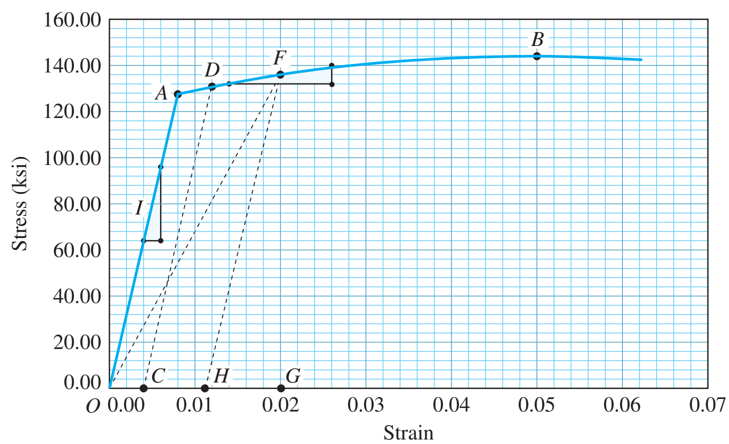


Figure 3.11 Stress–strain curve for Example 3.1.

(a) Point A is the proportional limit in Figure 3.11. The stress at point A is:

ANS. $\sigma_{prop} = 128$ ksi.

(b) The stress at point B in Figure 3.11 is the ultimate as it is largest stress on the stress–strain curve.

ANS. $\sigma_{\text{ult}} = 144$ ksi.

(c) The offset strain of 0.004 (or 0.4%) corresponds to point C. We can draw a line parallel to OA from point C, which intersects the stress–strain curve at point D. The stress at point D is the offset yield stress

ANS. $\sigma_{\text{yield}} = 132$ ksi.

(d) The modulus of elasticity E is the slope of line OA. Using the triangle at point I we can find E,

$$E = \frac{96\text{ ksi} - 64\text{ ksi}}{0.006 - 0.004} = 16(10^3)\text{ ksi} \tag{E1}$$

ANS. $E = 16{,}000\text{ ksi}$

(e) At point F the stress is 136 ksi. We can find the tangent modulus by finding the slope of the tangent at F,

$$E_t = \frac{140\text{ ksi} - 132\text{ ksi}}{0.026 - 0.014} = 666.67\text{ ksi} \tag{E2}$$

ANS. $E_t = 666.7\text{ ksi}$

(f) We can use triangle OFG to calculate the slope of OF to obtain secant modulus of elasticity at 136 ksi.

$$E_s = \frac{136\text{ ksi} - 0}{0.02 - 0} = 6800\text{ ksi} \tag{E3}$$

ANS. $E_s = 6800\text{ ksi}$

(g) To find the plastic strain at 136 ksi, we draw a line parallel to OA through point F. Following the description in Figure 3.4, OH represents the plastic strain. We know that the value of plastic strain will be between 0.01 and 0.012. We can do a more accurate calculation by noting that the plastic strain OH is the total strain OG minus the elastic strain HG. We find the elastic strain by dividing the stress at F (136 ksi) by the modulus of elasticity E:

$$\varepsilon_{\text{plastic}} = \varepsilon_{\text{total}} - \varepsilon_{\text{elastic}} = 0.02 - \frac{136\text{ ksi}}{16{,}000\text{ ksi}} = 0.0115 \tag{E4}$$

ANS. $\varepsilon_{\text{plastic}} = 11{,}500\ \mu$

3.1.4* Strain Energy

In the design of springs and dampers, the energy stored or dissipated is as significant as the stress and deformation. In designing automobile structures for crash worthiness, for example, we must consider how much kinetic energy is dissipated through plastic deformation. Some failure theories too, are based on energy rather than on maximum stress or strain. Minimum-energy principles are thus an important alternative to equilibrium equations and can often simplify our calculation.

The energy stored in a body due to deformation is the **strain energy,** U, and the strain energy per unit volume is the **strain energy density**, U_0:

$$U = \int_V U_0\, dV \tag{3.4}$$

where V is the volume of the body. Geometrically, U_0 is the area underneath the stress–strain curve up to the point of deformation. From Figure 3.12,

$$U_0 = \int_0^{\varepsilon} \sigma\, d\varepsilon \tag{3.5}$$

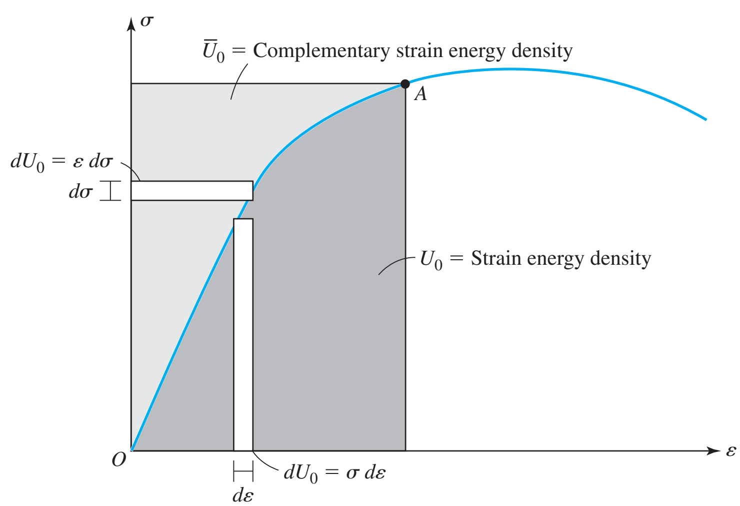


Figure 3.12 Energy densities.

The strain energy density has the same dimensions as stress since strain is dimensionless, but the units of strain energy density are different —$N \cdot m/m^3$, J/m^3, $in. \cdot lb/in.^3$, or $ft \cdot lb/ft^3$. Figure 3.12 also shows the **complementary strain energy density** $\bar{U}_0$, defined as

$$\bar{U}_0 = \int_0^{\sigma} \varepsilon \, d\sigma \tag{3.6}$$

The strain energy density at the yield point is called **modulus of resilience** (Figure 3.13a). This property is a measure of the recoverable (elastic) energy per unit volume that can be stored in a material. Since a spring is designed to operate in the elastic range, the higher the modulus of resilience, the more energy it can store.

The strain energy density at rupture is called **modulus of toughness**. This property is a measure of the energy per unit volume that can be absorbed by a material without breaking and is important in resistance to cracks and crack propagation. Whereas a *strong* material has high ultimate stress, a *tough* material has large area under the stress–strain curve, as seen in Figure 3.13*c*. It should be noted that strain energy density, complementary strain energy density, modulus of resilience, and modulus of toughness all have units of energy per unit volume.

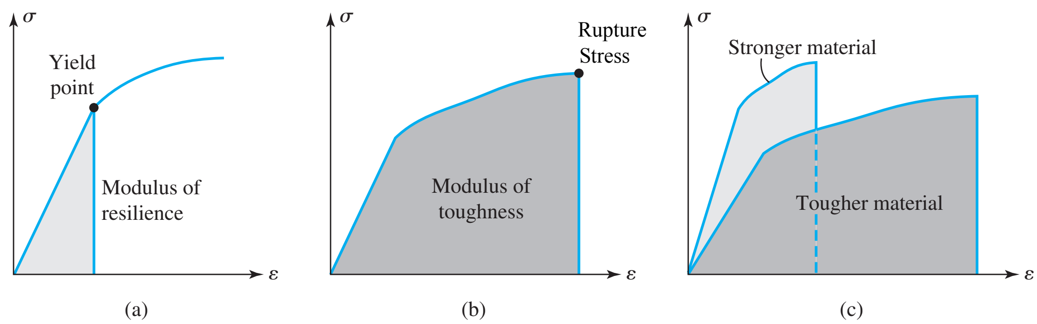


Figure 3.13 Energy-related moduli.

Linear Strain Energy Density

Most engineering structures are designed to function without permanent deformation. Thus most of the problems we will work with involve linear–elastic material. *Normal* stress and strain in the linear region are related by Hooke's law. Substituting $\sigma = E\varepsilon$ in Equation (3.5) and integrating, we obtain $U_0 = \int_0^{\varepsilon} E\varepsilon \, d\varepsilon = E\varepsilon^2/2$, which, again using Hooke's law, can be rewritten as

$$U_0 = \frac{1}{2}\sigma\varepsilon \tag{3.7}$$

Equation (3.7) reflects that the strain energy density is equal to the area of the triangle underneath the stress–strain curve in the linear region. Similarly, Equation (3.8) can be written using the *shear* stress–strain curve:

$$U_0 = \frac{1}{2}\tau\gamma \tag{3.8}$$

Strain energy, and hence strain energy density, is a scalar quantity. We can add the strain energy density due to the individual stress and strain components to obtain

$$U_0 = \frac{1}{2}[\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}] \tag{3.9}$$

EXAMPLE 3.2

For the titanium alloy in Example 3.1, determine: (a) The modulus of resilience. Use proportional limit as an approximation for yield point. (b) Strain energy density at a stress level of 136 ksi. (c) Complementary strain energy density at a stress level of 136 ksi. (d) Modulus of toughness.

PLAN

We can identify the proportional limit, the point on curve with stress of 136 ksi and the rupture point and calculate the areas under the curve to obtain the quantities of interest.

SOLUTION

Figure 3.11 is redrawn as Figure 3.14.
Point *A* is the proportional limit we can use to approximate the yield point in Figure 3.14. The area of the triangle OAA_1 can be calculated as shown in Equation (E1) and equated to modulus of resilience.

$$AOA_1 = \frac{128 \times 0.008}{2} = 0.512 \qquad \text{(E1)}$$

ANS. The modulus of resilience is 0.512 in.·kips/in.³.

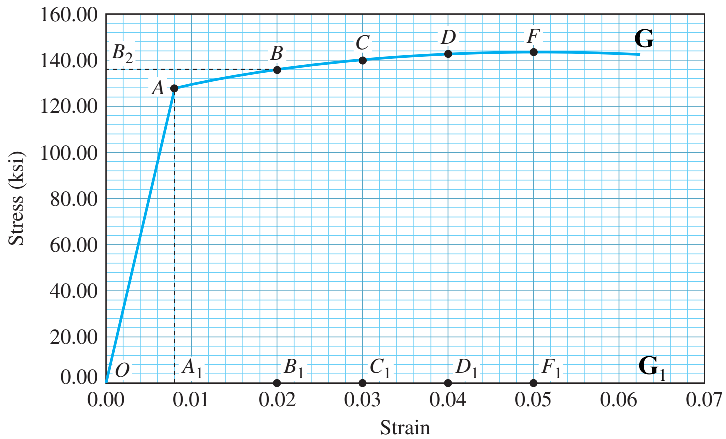


Figure 3.14 Area under curve in Example 3.2.

Point *B* in Figure 3.14 is at 136 ksi. The strain energy density at point *B* is the area AOA_1 plus the area AA_1BB_1. The area AA_1BB_1 can be approximated as the area of a trapezoid and found as

$$AA_1BB_1 = \frac{(128 + 136)\,0.012}{2} = 1.584 \qquad \text{(E2)}$$

The strain energy density at *B* (136 ksi) is $U_B = 0.512 + 1.584$

ANS. $U_B = 2.1 \text{ in.} \cdot \text{kips/in.}^3$

The complementary strain energy density at *B* can be found by subtracting U_B from the area of the rectangle OB_2BB_1. Thus, $\bar{U}_B = 136 \times 0.02 - 2.1$.

ANS. $\bar{U}_B = 0.62 \text{ in.} \cdot \text{kips/in.}^3$

The rupture stress corresponds to point *G* on the graph. The area underneath the curve in Figure 3.14 can be calculated by approximating the curve as a series of straight lines *AB*, *BC*, *CD*, *DF*, and *FG*.

$$BB_1CC_1 = \frac{(136 + 140)\,0.010}{2} = 1.38 \qquad \text{(E3)}$$

$$CC_1DD_1 = \frac{(140 + 142)\,0.010}{2} = 1.41 \qquad \text{(E4)}$$

$$DD_1FF_1 = \frac{(142 + 144)\,0.010}{2} = 1.43 \qquad \text{(E5)}$$

$$FF_1GG_1 = \frac{(144 + 142)\,0.012}{2} = 1.716 \qquad \text{(E6)}$$

The total area is the sum of the areas given by Equations (E1) through (E6), or 8.032.

ANS. The modulus of toughness is 8.03 in.·kips/in.³.

COMMENTS

1. Approximation of the curve by a straight line for the purpose of finding areas is the same as using the trapezoidal rule of integration.

2. In Table 3.3, there were many data points between the points shown by letters *A* through *G* in Figure 3.14. We can obtain more accurate results if we approximate the curve between two data points by a straight line. This would become tedious unless we use a spread sheet as discussed in Appendix B.1.

3.2 THE LOGIC OF THE MECHANICS OF MATERIALS

We now have all the pieces in place for constructing the logic that is used for constructing theories and obtain formulas for the simplest one-dimensional structural members, such as in this book, to linear or nonlinear structural members of plates and shells seen in graduate courses. In Chapter 1 we studied the two steps of relating stresses to internal forces and relating internal forces to external forces. In Chapter 2 we studied the relationship of strains and displacements. Finally, in Section 3.1 we studied the relationship of stresses and strains. In this section we integrate all these concepts, to show the logic of structural analysis.

Figure 3.15 shows how we relate displacements to external forces. It is possible to start at any point and move either clockwise (shown by the filled arrows ⟶) or counterclockwise (shown by the hollow arrows ⟶). No one arrow directly relates displacement to external forces, because we cannot relate the two without imposing limitations and making assumptions regarding the geometry of the body, material behavior, and external loading.

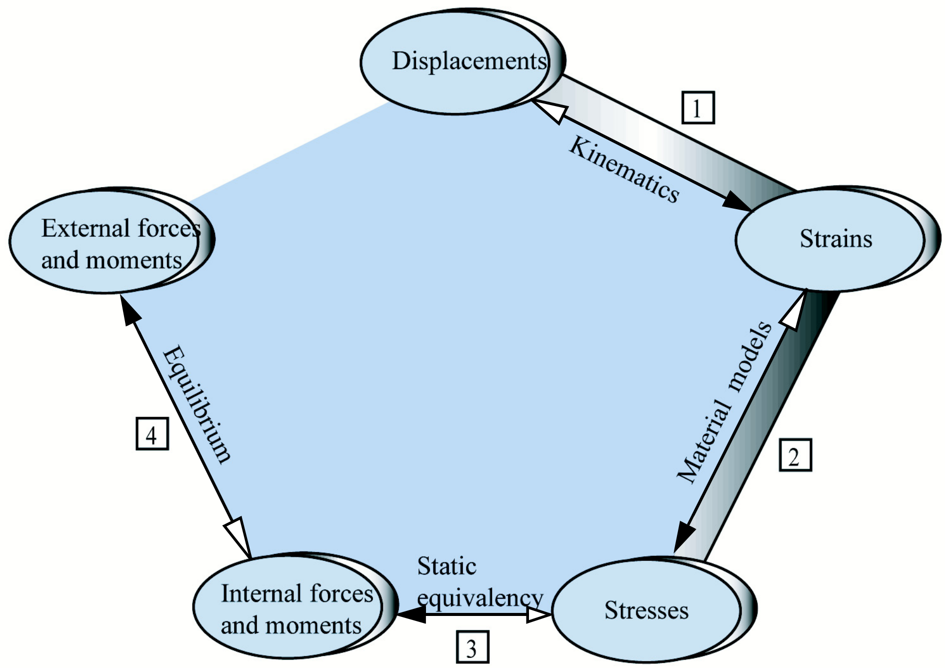


Figure 3.15 Logic in structural analysis.

The starting point in the logical progression depends on the information we have or can deduce about a particular variable. If the material model is simple, then it is possible to deduce the behavior of stresses, as we did in Chapter 1. But as the complexity in material models grows, so does the complexity of stress distributions, and deducing stress distribution becomes increasingly difficult. Unlike stresses, displacements can be measured directly or observed or deduced from geometric considerations. Later chapters will develop theories for axial rods, torsion of shafts, and bending of beams by approximating displacements and relating these displacements to external forces and moments using the logic shown in Figure 3.15.

Examples 3.3 and 3.4 demonstrate logic of problem solving shown in Figure 3.15. Its modular character permits the addition of complexities without changing the logical progression of derivation, as demonstrated by Example 3.5.

EXAMPLE 3.3

A rigid plate is attached to two 10 mm × 10 mm square bars (Figure 3.16). The bars are made of hard rubber with a shear modulus G = 1.0 MPa. The rigid plate is constrained to move horizontally due to action of the force F. If the horizontal movement of the plate is 0.5 mm, determine the force F assuming uniform shear strain in each bar.

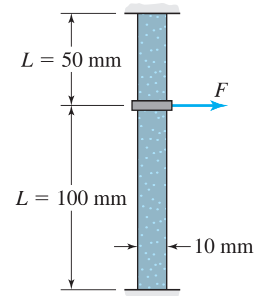


Figure 3.16 Geometry in Example 3.3.

PLAN

We can draw an approximate deformed shape and calculate the shear strain in each bar. Using Hooke's law we can find the shear stress in each bar. By multiplying the shear stress by the area we can find the equivalent internal shear force. By drawing the free-body diagram of the rigid plate we can relate the internal shear force to the external force F and determine F.

SOLUTION

1. *Strain calculation:* Figure 3.17*a* shows an approximate deformed shape. Assuming small strain we can find the shear strain in each bar:

$$\tan\gamma_{AB} \approx \gamma_{AB} = \frac{0.5\text{ mm}}{100\text{ mm}} = 5000\ \mu\text{rad} \tag{E1}$$

$$\tan\gamma_{CD} \approx \gamma_{CD} = \frac{0.5\text{ mm}}{50\text{ mm}} = 10{,}000\ \mu\text{rad} \tag{E2}$$

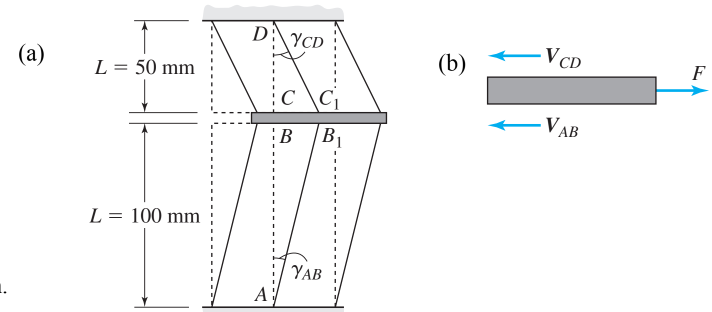


Figure 3.17 (a) Deformed geometry. (b) Free-body diagram.

2. *Stress calculation:* From Hooke's law $\tau = G\gamma$ we can find the shear stress in each bar:

$$\tau_{AB} = (10^6\text{ N/m}^2)(5000)(10^{-6}) = 5000\text{ N/m}^2 \tag{E3}$$

$$\tau_{CD} = (10^6\text{ N/m}^2)(10{,}000)(10^{-6}) = 10{,}000\text{ N/m}^2 \tag{E4}$$

3. *Internal force calculation:* The cross-sectional area of the bar is $A = 100\text{ mm}^2 = 100(10^{-6})\text{ m}^2$. Assuming uniform shear stress, we can find the shear force in each bar:

$$V_{AB} = \tau_{AB}A = (5000\text{ N/m}^2)(100)(10^{-6})\text{ m}^2 = 0.5\text{ N} \tag{E5}$$

$$V_{CD} = \tau_{CD}A = (10{,}000\text{ N/m}^2)(100)(10^{-6})\text{ m}^2 = 1.0\text{ N} \tag{E6}$$

4. *External force calculation:* We can make imaginary cuts on either side of the rigid plate and draw the free-body diagram as shown in Figure 3.17*b*. From equilibrium of the rigid plate we can obtain the external force F as

$$F = V_{AB} + V_{CD} = 1.5\text{ N} \tag{E7}$$

ANS. $F = 1.5$ N

EXAMPLE 3.4*

The steel bars (E = 200 GPa) in the truss shown in Figure 3.18 have cross-sectional area of 100 mm^2. Determine the forces F_1 and F_2 if the displacements u and v of the pins in the x and y directions, respectively, are as given below.

$$u_B = -0.500 \text{ mm} \qquad v_B = -2.714 \text{ mm}$$
$$u_C = -1.000 \text{ mm} \qquad v_C = -6.428 \text{ mm}$$
$$u_D = 1.300 \text{ mm} \qquad v_D = -2.714 \text{ mm}$$

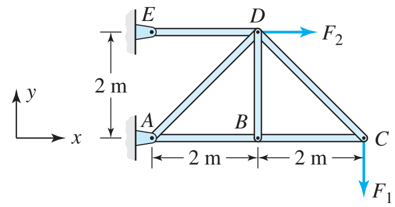


Figure 3.18 Pin displacements in Example 3.4.

PLAN

We can find strains using small-strain approximation as in Example 2.8. Following the logic in Figure 3.15 we can find stresses and then the internal force in each member. We can then draw free-body diagrams of joints C and D to find the forces F_1 and F_2.

SOLUTION

1. *Strain calculations:* The strains in the horizontal and vertical members can be found directly from the displacements,

$$\varepsilon_{AB} = \frac{u_B - u_A}{L_{AB}} = -0.250(10^{-3}) \text{ m/m} \qquad \varepsilon_{BC} = \frac{u_C - u_B}{L_{BC}} = -0.250(10^{-3}) \text{ m/m}$$
$$\varepsilon_{ED} = \frac{u_D - u_E}{L_{ED}} = 0.650(10^{-3}) \text{ m/m} \qquad \varepsilon_{BD} = \frac{v_D - v_B}{L_{BD}} = 0 \tag{E1}$$

For the inclined member AD we first find the relative displacement vector $\bar{\mathbf{D}}_{AD}$ and then take a dot product with the unit vector $\bar{\mathbf{i}}_{AD}$, to obtain the deformation of AD as

$$\bar{\mathbf{D}}_{AD} = (u_D\bar{\mathbf{i}} + v_D\bar{\mathbf{j}}) - (u_A\bar{\mathbf{i}} + v_A\bar{\mathbf{j}}) = (1.3\bar{\mathbf{i}} - 2.714\bar{\mathbf{j}}) \text{ mm}$$
$$\bar{\mathbf{i}}_{AD} = \cos 45\ \bar{\mathbf{i}} + \sin 45\ \bar{\mathbf{j}} = 0.707\bar{\mathbf{i}} + 0.707\bar{\mathbf{j}} \tag{E2}$$

$$\delta_{AD} = \bar{\mathbf{D}}_{AD} \cdot \bar{\mathbf{i}}_{AD} = (1.3 \text{ mm})(0.707) + (-2.714 \text{ mm})(0.707) = -1.000 \text{ mm} \tag{E3}$$

The length of AD is $L_{AD} = 2.828$ m we obtain the strain in AD as

$$\varepsilon_{AD} = \frac{\delta_{AD}}{L_{AD}} = \frac{-1.000(10^{-3}) \text{ m}}{2.828 \text{ m}} = -0.3535(10^{-3}) \text{ m/m} \tag{E4}$$

Similarly for member CD we obtain

$$\bar{\mathbf{D}}_{CD} = (u_D\bar{\mathbf{i}} + v_D\bar{\mathbf{j}}) - (u_C\bar{\mathbf{i}} + v_C\bar{\mathbf{j}}) = (2.3\ \bar{\mathbf{i}} + 3.714\bar{\mathbf{j}}) \text{ mm}$$
$$\bar{\mathbf{i}}_{CD} = -\cos 45\ \bar{\mathbf{i}} + \sin 45\ \bar{\mathbf{j}} = -0.707\bar{\mathbf{i}} + 0.707\bar{\mathbf{j}} \tag{E5}$$

$$\delta_{CD} = \bar{\mathbf{D}}_{CD} \cdot \bar{\mathbf{i}}_{CD} = (2.3 \text{ mm})(-0.707) + (3.714 \text{ mm})(0.707) = 1.000 \text{ mm} \tag{E6}$$

The length of CD is $L_{CD} = 2.828$ m and we obtain the strain in CD as

$$\varepsilon_{CD} = \frac{\delta_{CD}}{L_{CD}} = \frac{1.000(10^{-3}) \text{ m}}{2.828 \text{ m}} = 0.3535(10^{-3}) \text{ m/m} \tag{E7}$$

2. *Stress calculations:* From Hooke's law $\sigma = E\varepsilon$, we can find stresses in each member:

$$\sigma_{AB} = (200 \times 10^9 \text{N/m}^2)(-0.250 \times 10^{-3}) = 50 \text{ MPa } (C)$$
$$\sigma_{BC} = (200 \times 10^9 \text{N/m}^2)(-0.250 \times 10^{-3}) = 50 \text{ MPa } (C)$$
$$\sigma_{ED} = (200 \times 10^9 \text{N/m}^2)(0.650 \times 10^{-3}) = 130 \text{ MPa} (T)$$
$$\sigma_{BD} = (200 \times 10^9 \text{N/m}^2)(0.000 \times 10^{-3}) = 0 \tag{E8}$$
$$\sigma_{AD} = (200 \times 10^9 \text{N/m}^2)(-0.3535 \times 10^{-3}) = 70.7 \text{ MPa } (C)$$
$$\sigma_{CD} = (200 \times 10^9 \text{N/m}^2)(0.3535 \times 10^{-3}) = 70.7 \text{ MPa } (T)$$

3. *Internal force calculations:* The internal normal force can be found from $N = \sigma A$, where the cross-sectional area is $A = 100 \times 10^{-6}$ m^2. This yields the following internal forces:

$$N_{AB} = 5\text{ kN (C)} \qquad N_{BC} = 5\text{ kN (C)}$$
$$N_{ED} = 13.0\text{ kN (T)} \qquad N_{BD} = 0 \qquad \text{(E9)}$$
$$N_{AD} = 7.07\text{ kN (C)} \qquad N_{CD} = 7.07\text{ kN (T)}$$

4. *External forces:* We draw free-body diagrams of pins *C* and *D* as shown in Figure 3.19.

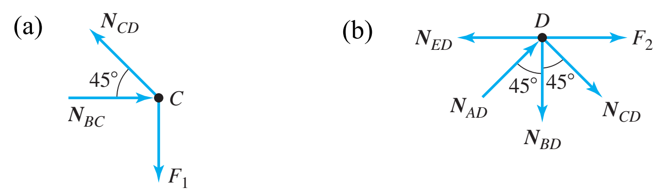


Figure 3.19 Free-body diagram of joint (a) *C* (b) *D*.

By equilibrium of forces in *y* direction in Figure 3.19*a*

$$N_{CD}\sin 45° - F_1 = 0 \qquad \text{(E10)}$$

ANS. $F_1 = 5\text{ kN}$

By equilibrium of forces in *x* direction in Figure 3.19*b*

$$F_2 + N_{CD}\sin 45° + N_{AD}\sin 45° - N_{ED} = 0 \qquad \text{(E11)}$$

ANS. $F_2 = 3\text{ kN}$

COMMENTS

1. Notice the direction of the internal forces. Forces that are pointed into the joint are compressive and the forces pointed away from the joint are tensile.
2. We used force equilibrium in only one direction to determine the external forces. We can use the equilibrium in the other direction to check our results. By equilibrium of forces in the x-direction in Figure 3.19a we obtain:

$$N_{BC} = = N_{CD}\cos 45° = 7.07\text{ kN}\cos 45° = 5\text{ kN}$$

which checks with the value we calculated. The forces in the *y* direction in Figure 3.19*b* must also be in equilibrium. With N_{BD} equal to zero we obtain N_{AD} should be equal to N_{CD}, which checks with the values calculated.

EXAMPLE 3.5

A canoe on top of a car is tied down using rubber stretch cords, as shown in Figure 3.20*a*. The undeformed length of the stretch cord is 40 in. The initial diameter of the cord is $d = 0.5$ in. and the modulus of elasticity of the cord is $E = 510$ psi. Assume that the path of the stretch cord over the canoe can be approximated as shown in Figure 3.20*b*. Determine the approximate force exerted by the cord on the carrier of the car.

(a)

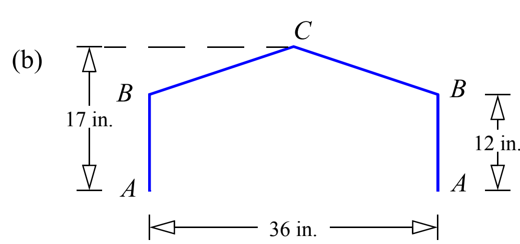


Figure 3.20 Approximation of stretch cord path on top of canoe in Example 3.5.

PLAN

We can find the stretched length L_f of the cord from geometry. Knowing L_f and $L_0 = 40$ in., we can find the average normal strain in the cord from Equation (2.1). Using the modulus of elasticity, we can find the average normal stress in the cord from Hooke's law, given by Equation (3.1). Knowing the diameter of the cord, we can find the cross-sectional area of the cord and multiply it by the normal stress to obtain the tension in the cord. If we make an imaginary cut in the cord just above *A*, we see that the tension in the cord is the force exerted on the carrier.

SOLUTION

1. *Strain calculations:* We can find the length *BC* using Figure 3.21a from the Pythagorean theorem:

$$BC = \sqrt{(5\text{ in.})^2 + (18\text{ in.})^2} = 18.68\text{ in.} \tag{E1}$$

Noting the symmetry, we can find the total length L_f of the stretched cord and the average normal strain:

$$L_f = 2(AB + BC) = 61.36\text{ in.} \tag{E2}$$

$$\varepsilon = \frac{L_f - L_0}{L_0} = \frac{61.36\text{ in.} - 40\text{ in.}}{40\text{ in.}} = 0.5341\text{ in./in.} \tag{E3}$$

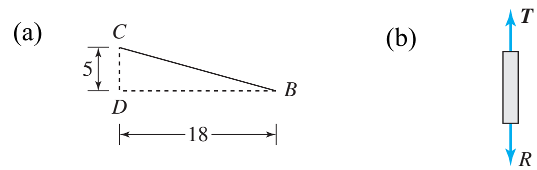


Figure 3.21 Calculations in Example 3.5 of (a) length (b) reaction force

2. *Stress calculation:* From Hooke's law we can find the stress as

$$\sigma = E\varepsilon = (510\ psi)(0.5341) = 272.38\ psi \tag{E4}$$

3. *Internal force calculations:* We can find the cross-sectional area from the given diameter $d = 0.5$ in. and multiply it with the stress to obtain the internal tension,

$$A = \frac{\pi d^2}{4} = \frac{\pi(0.5\text{ in.})^2}{4} = 0.1963\text{ in.}^2 \tag{E5}$$

$$\boldsymbol{T} = \sigma A = (0.1963\text{ in.}^2)(272.38\ psi) = 53.5\ lb \tag{E6}$$

4. *Reaction force calculation:* We can make a cut just above A and draw the free-body diagram as shown in Figure 3.21*b* to calculate the force R exerted on the carrier,

$$R = \boldsymbol{T} \tag{E7}$$

ANS. $\boldsymbol{T} = 53.5\text{ lb}$

COMMENTS

1. Unlike in the previous two examples, where relatively accurate solutions would be obtained, in this example we have large strains and several other approximations, as elaborated in the next comment. The only thing we can say with some confidence is that the answer has the right order of magnitude.
2. The following approximations were made in this example:
(a) The path of the cord should have been an inclined straight line between the carrier rail and the point of contact on the canoe, and then the path should have been the contour of the canoe.
(b) The strain along the cord is nonuniform, which we approximated by a uniform average strain.
(c) The stress–strain curve of the rubber cord is nonlinear. Thus as the strain changes along the length, so does the modulus of elasticity E, and we need to account for this variation of E in the calculation of stress.
(d) The cross-sectional area for rubber will change significantly with strain and must be accounted for in the calculation of the internal tension.
3. Depending on the need of our accuracy, we can include additional complexities to address the error from the preceding approximations.
(a) Suppose we did a better approximation of the path as described in part (2a) but made no other changes. In such a case the only change would be in the calculation of L_f in Equation (E2) (see Problem 2.87), but the rest of the equations would remain the same.
(b) Suppose we make marks on the cord every 2 in. before we stretch it over the canoe. We can then measure the distance between two consecutive marks when the cord is stretched. Now we have L_f for each segment and can repeat the calculation for each segment (see Problem 3.68).
(c) Suppose, in addition to the above two changes, we have the stress–strain curve of the stretch cord material. Now we can use the tangent modulus in Hooke's law for each segment, and hence we can get more accurate stresses in each segment. We can then calculate the internal force as before (see Problem 3.69).
(d) Rubber has a Poisson's ratio of 0.5. Knowing the longitudinal strain from Equation (E3) for each segment, we can compute the transverse strain in each segment and find the diameter of the cord in the stretched position in each segment. This will give us a more accurate area of cross section, and hence a more accurate value of internal tension in the cord (see Problem 3.70).
4. These comments demonstrate how complexities can be added one at a time to improve the accuracy of a solution. In a similar manner, we shall derive theories for axial members, shafts, and beams in Chapters 4 through 6, to which complexities can be added as asked of you in "Stretch yourself" problems. Which complexity to include depends on the individual case and our need for accuracy.

Consolidate your knowledge

1. In your own words, describe the tension test and the quantities that can be calculated from the experiment.

QUICK TEST 3.1 **Time: 15 minutes/Total: 20 points**

Grade yourself using the answers given in Appendix E. Each question is worth two points.

1. What are the typical units of modulus of elasticity and Poisson's ratio in the metric system?
2. Define offset yield stress.
3. What is strain hardening?
4. What is necking?
5. What is the difference between proportional limit and yield point?
6. What is the difference between a brittle material and a ductile material?
7. What is the difference between linear material behavior and elastic material behavior?
8. What is the difference between strain energy and strain energy density?
9. What is the difference between modulus of resilience and modulus of toughness?
10. What is the difference between a strong material and a tough material?

3.3 FAILURE AND FACTOR OF SAFETY

There are many types of failures. The breaking of the ship *S.S. Schenectady* (Chapter 1) was a failure of strength, whereas the failure of the O-ring joints in the shuttle *Challenger* (Chapter 2) was due to excessive deformation. **Failure** implies that a component or a structure does not perform the function for which it was designed.

A machine component may interfere with other moving parts because of excessive deformation; a chair may feel rickety because of poor joint design; a gasket seal leaks because of insufficient deformation of the gasket at some points; lock washers may not deform enough to provide the spring force needed to keep bolted joints from becoming loose; a building undergoing excessive deformation may become aesthetically displeasing. These are examples of failure caused by too little or too much deformation.

The stiffness of a structural element depends on the modulus of elasticity of the material as well as on the geometric properties of the member, such as cross-sectional area, area moments of inertia, polar moments of inertia, and the length of the components. The use of carpenter's glue in the joints of a chair to prevent a rickety feeling is a simple example of increasing joint and structure stiffness by using adhesives.

Prevention of a component fracture is an obvious design objective based on strength. At other times, our design objective may be avoid to making a component *too* strong. The adhesive bond between the lid and a sauce bottle must break so that the bottle may be opened by hand; shear pins must break before critical components get damaged; the steering column of an automobile must collapse rather than impale the passenger in a crash. Ultimate normal stress is used for assessing failure due to breaking or rupture particularly for brittle materials.

Permanent deformation rather than rupture is another stress-based failure. Dents or stress lines in the body of an automobile; locking up of bolts and screws because of permanent deformation of threads; slackening of tension wires holding a structure in place—in each of these examples, plastic deformation is the cause of failure. Yield stress is used for assessing failure due to plastic deformation, particularly for ductile materials.

A support in a bridge may fail, but the bridge can still carry traffic. In other words, the failure of a component does not imply failure of the entire structure. Thus the strength of a structure, or the deflection of the entire structure, may depend on a large number of variables. In such cases loads on the structure are used to characterize failure. Failure loads may be based on the stiffness, the strength, or both.

A margin of safety must be built into any design to account for uncertainties or a lack of knowledge, lack of control over the environment, and the simplifying assumptions made to obtain results. The measure of this margin of safety is the factor of safety K_{safety} defined as

$$K_{safety} = \frac{\text{failure-producing value}}{\text{computed (allowable) value}} \tag{3.10}$$

Equation (3.10) implies that the factor of safety must always be *greater than 1*. The numerator could be the failure deflection, failure stress, or failure load and is assumed known. In analysis, the denominator is determined, and from it the factor of safety is found. In design, the factor of safety is specified, and the variables affecting the denominator are determined such that the denominator value is not exceeded. Thus in design the denominator is often referred to as the allowable value.

Several issues must be considered in determining the appropriate factor of safety in design. No single issue dictates the choice. The value chosen is a compromise among various issues and is arrived at from experience.

Material or operating costs are the primary reason for using a low factor of safety, whereas liability cost considerations push for a greater factor of safety. A large fixed cost could be due to expensive material, or due to large quantity of material used to meet a given factor of safety. Greater weight may result in higher fuel costs. In the aerospace industries the operating costs supersede material costs. Material costs dominate the furniture industry. The automobile industry seeks a compromise between fixed and running costs. Though liability is a consideration in all design, the building industry is most conscious of it in determining the factor of safety.

Lack of control or lack of knowledge of the operating environment also push for higher factors of safety. Uncertainties in predicting earthquakes, cyclones, or tornadoes, for examples, require higher safety factors for the design of buildings located in regions prone to these natural calamities. A large scatter in material properties, as usually seen with newer materials, is another uncertainty pushing for higher factor of safety.

Human safety considerations not only push the factor of safety higher but often result in government regulations of the factors of safety, as in building codes.

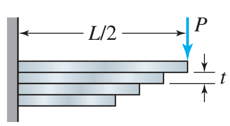
This list of issues affecting the factor of safety is by no means complete, but is an indication of the subjectivity that goes into the choice of the factor of safety. The factors of safety that may be recommended for most applications range from 1.1 to 6.

EXAMPLE 3.6

In the leaf spring design in Figure 3.22 the formulas for the maximum stress σ and deflection δ given in Equation (3.11) are derived from theory of bending of beams (see Example 7.4):

$$\sigma = \frac{3PL}{nbt^2} \qquad \delta = \frac{3PL^3}{4Enbt^3} \tag{3.11}$$

where P is the load supported by the spring, L is the length of the spring, n is the number of leaves b is the width of each leaf, t is the thickness of each leaf, and E is the modulus of elasticity. A spring has the following data: $L = 20$ in., $b = 2$ in., $t = 0.25$ in., and $E = 30{,}000$ ksi. The failure stress is $\sigma_{failure} = 120$ ksi, and the failure deflection is $\delta_{failure} = 0.5$ in. The spring is estimated to carry a maximum force P = 250 lb and is to have a factor of safety of $K_{safety} = 4$. (a) Determine the minimum number of leaves. (b) For the answer in part (a) what is the real factor of safety?



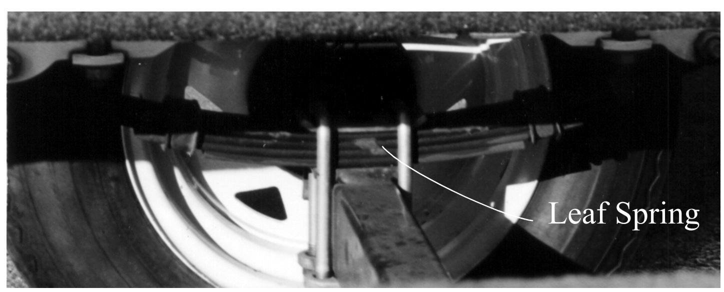


Figure 3.22 Leaf spring in Example 3.6.

PLAN

(a) The allowable stress and allowable deflection can be found from Equation (3.10) using the factor of safety of 4. Equation (3.11) can be used with two values of n to ensure that the allowable values of stress and deflection are not exceeded. The higher of the two values of n is the minimum number of leaves in the spring design. (b) Substituting n in Equation (3.11), we can compute the maximum stress and deflection and obtain the two factors of safety from Equation (3.10). The lower value is the real factor of safety.

SOLUTION

(a) The allowable values for stress and deflection can be found from Equation (3.10) as:

$$\sigma_{allow} = \frac{\sigma_{failure}}{K_{safety}} = \frac{120 \text{ ksi}}{4} = 30 \text{ ksi} \tag{E1}$$

$$\delta_{allow} = \frac{\delta_{failure}}{K_{safety}} = \frac{0.5 \text{ in.}}{4} = 0.125 \text{ in.} \tag{E2}$$

Substituting the given values of the variables in the stress formula in Equation (3.11), we obtain the maximum stress, which should be less than the allowable stress. From this we can obtain one limitation on n:

$$\sigma = \frac{3PL}{nbt^2} = \frac{3(250 \text{ lb})(20 \text{ in.})}{n(2 \text{ in.})(0.25 \text{ in.})^2} = \frac{120(10^3) \text{ psi}}{n} \le 30(10^3) \text{ psi} \qquad \text{or} \tag{E3}$$

$$n \ge 4 \tag{E4}$$

Substituting the given values in the deflection formula, in Equation (3.11), we obtain the maximum deflection, which should be less than the allowable stress we thus obtain one limitation on n:

$$\delta = \frac{3PL^3}{4Enbt^3} = \frac{3(250 \text{ lb})(20 \text{ in.})^3}{4(30 \times 10^6 \text{ psi})(n)(2 \text{ in.})(0.25 \text{ in.})^3} = \frac{1.6 \text{ in.}}{n} \le 0.125 \text{ in.} \qquad \text{or} \tag{E5}$$

$$n \ge 12.8 \tag{E6}$$

The minimum number of leaves that will satisfy Equations (E4) and (E6) is our answer.

ANS. $n = 13$

(b) Substituting $n = 13$ in Equations (E3) and (E5) we find the computed values of stress and deflection and the factors of safety from Equation (3.10).

$$\sigma_{comp} = \frac{120(10^3) \text{ psi}}{13 \text{ psi}} = 9.23(10^3) \text{ psi} \qquad K_\sigma = \frac{\sigma_{failure}}{\sigma_{comp}} = \frac{120(10^3) \text{ psi}}{9.23(10^3) \text{ psi}} = 13 \tag{E7}$$

$$\delta_{comp} = \frac{1.6 \text{ in.}}{13} = 0.1232 \text{ in} \qquad K_\delta = \frac{\delta_{failure}}{\delta_{comp}} = \frac{0.5 \text{ in.}}{0.1232 \text{ in.}} = 4.06 \tag{E8}$$

The factor of safety for the system is governed by the lowest factor of safety, which in our case is given by Equation (E8).

ANS. $K_\delta = 4.06$

COMMENTS

1. This problem demonstrates the difference between the allowable values, which are used in design decisions based on a specified factor of safety, and computed values, which are used in analysis for finding the factor of safety.
2. For purposes of design, formulas are initially obtained based on simplified models, as shown in Figure 3.22. Once the preliminary relationship between variables has been established, then complexities are often incorporated by using factors determined experimentally. Thus the deflection of the spring, accounting for curvature, end support, variation of thickness, and so on is given by $\delta = K(3PL^3/4Enbt^3)$, where K is determined experimentally as function of the complexities not accounted for in the simplified model. This comment highlights how the mechanics of materials provides a guide to developing formulas for complex realities.

PROBLEM SET 3.1

Stress–strain curves

3.1 -3.5 *A tensile test specimen having a diameter of 10 mm and a gage length of 50 mm was tested to fracture. The stress–strain curve from the tension test is shown in Figure P3.3. The lower plot is the expanded region OAB and associated with the strain values given on the lower scale. Solve Problems 3.1 through 3.5.*

3.1 Determine (a) the ultimate stress; (b) the fracture stress; (c) the modulus of elasticity; (d) the proportional limit; (e) the offset yield stress at 0.2%; (f) the tangent modulus at stress level of 420 MPa; (g) the secant modulus at stress level of 420 MPa.

3.2 Determine the axial force acting on the specimen when it is extended by (a) 0.2 mm; (b) 4.0 mm.

3.3 Determine the extension of the specimen when the axial force on the specimen is 33 kN.

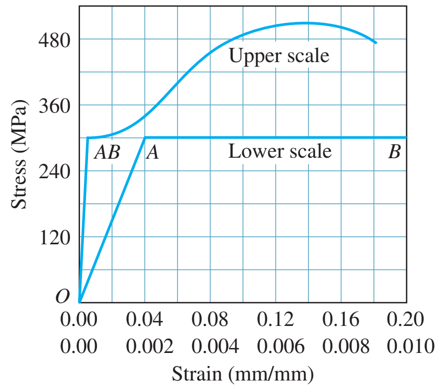


Figure P3.3

3.4 Determine the total strain, the elastic strain, and the plastic strain when the axial force on the specimen is 33 kN.

3.5 After the axial load was removed, the specimen was observed to have a length of 54 mm. What was the maximum axial load applied to the specimen?

3.6—3.10 A tensile test specimen having a diameter of $\frac{5}{8}$ in. and a gage length of 2 in. was tested to fracture. The stress–strain curve from the tension test is shown in Figure P3.6. The lower plot is the expanded region OAB and associated with the strain values given on the lower scale. Solve Problems 3.6 through 3.10 using this graph.

3.6 Determine (a) the ultimate stress; (b) the fracture stress; (c) the modulus of elasticity; (d) the proportional limit.(e)the offset yield stress at 0.1%; (f) the tangent modulus at the stress level of 72 kips; (g) the secant modulus at the stress level of 72 kips.

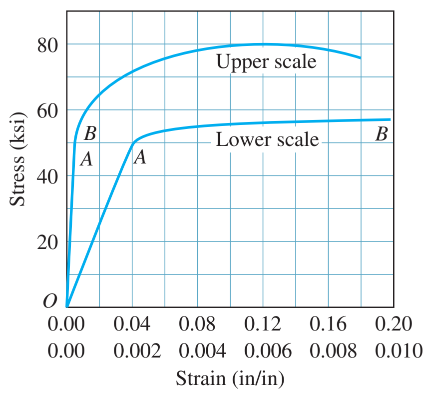


Figure P3.6

3.7 Determine the axial force acting on the specimen when it is extended by (a) 0.006 in.; (b) 0.120 in.

3.8 Determine the extension of the specimen when the axial force on the specimen is 20 kips.

3.9 Determine the total strain, the elastic strain, and the plastic strain when the axial force on the specimen is 20 kips.

3.10 After the axial load was removed, the specimen was observed to have a length of 2.12 in. What was the maximum axial load applied to the specimen?

3.11 A typical stress-strain graph for cortical bone is shown in Figure P3.11. Determine (a) the modulus of elasticity; (b) the proportional limit; (c) the yield stress at 0.15% offset; (d) the secant modulus at stress level of 130 MPa; (d) the tangent modulus at stress level of 130 MPa; (e) the permanent strain at stress level of 130MPa. (f) If the shear modulus of the bone is 6.6 GPa, determine Poisson's ratio assuming the bone is isotropic. (g) Assuming the bone specimen was 200 mm long and had a material cross-sectional area of 250 mm^2, what is the elongation of the bone when a 20-kN force is applied?

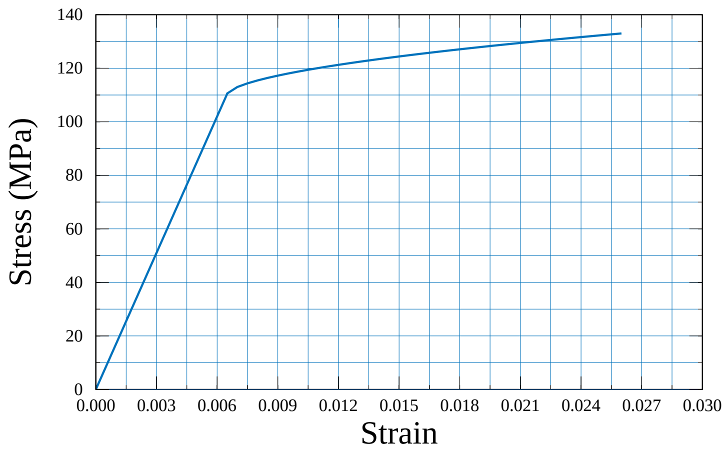


Figure P3.11

3.12 A 12 mm × 12 mm square metal alloy having a gage length of 50 mm was tested in tension. The results are given in Table P3.12. Draw the stress–strain curve and calculate the following quantities. (a) the modulus of elasticity. (b) the proportional limit. (c) the yield stress at 0.2% offset. (d) the tangent modulus at a stress level of 1400 MPa. (e) the secant modulus at a stress level of 1400 MPa. (f) the plastic strain at a stress level of 1400 MPa. (Use of a spreadsheet is recommended.)

TABLE P3.12

Load (kN)	Change in Length (mm)	Load (kN)	Change in Length (mm)
0.00	0.00	200.01	5.80
17.32	0.02	204.65	7.15
60.62	0.07	209.99	8.88
112.58	0.13	212.06	9.99
147.22	0.17	212.17	11.01
161.18	0.53	208.64	11.63
168.27	1.10	204.99	12.03
176.03	1.96	199.34	12.31
182.80	2.79	192.15	12.47
190.75	4.00	185.46	12.63
193.29	4.71	Break	

3.13 A mild steel specimen of 0.5 in. diameter and a gage length of 2 in. was tested in tension. The test results are reported Table P3.13. Draw the stress–strain curve and calculate the following quantities: (a) the modulus of elasticity; (b) the proportional limit; (c) the yield stress at 0.05% offset; (d) the tangent modulus at a stress level of 50 ksi; (e) the secant modulus at a stress level of 50 ksi; (f) the plastic strain at a stress level of 50 ksi. (Use of a spreadsheet is recommended.)

TABLE P3.13

Load (10^3 lb)	Change in Length (10^{-3} in.)	Load (10^3 lb)	Change in Length (10^{-3} in.)
0.00	0.00	11.18	112.10
3.11	1.28	11.72	140.40
7.24	2.96	11.99	161.21
7.50	3.06	12.27	192.65
7.70	8.76	12.41	214.22
7.90	19.05	12.55	245.93
8.16	28.70	12.70	283.47
8.46	37.73	12.77	316.36
8.82	47.18	12.84	363.10
9.32	59.06	12.04	385.34
9.86	70.85	11.44	396.03
10.40	84.23	10.71	406.42
10.82	97.85	9.96	414.72
		Break	

3.14 A rigid bar AB of negligible weight is supported by cable of diameter 1/4 in, as shown in Figure P3.14. The cable is made from a material that has a stress- strain curve shown in Figure P3.6. (a) Determine the extension of the cable when P = 2 kips. (b) What is the permanent deformation in BC when the load P is removed?

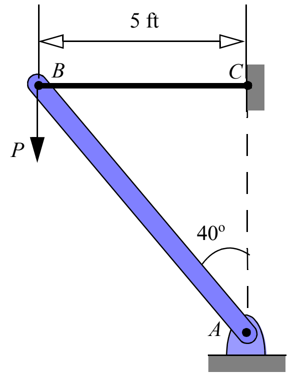


Figure P3.14

3.15 A rigid bar AB of negligible weight is supported by cable of diameter 1/4 in., as shown in Figure P3.14. The cable is made from a material that has a stress-strain curve shown in Figure P3.6. (a) Determine the extension of the cable when P =4.25 kips. (b) What is the permanent deformation in the cable when the load P is removed?

Material constants

3.16 A rectangular bar has a cross-sectional area of 2 in.^{2} and an undeformed length of 5 in., as shown in Figure 3.18. When a load P = 50,000 lb is applied, the bar deforms to a position shown by the colored shape. Determine the modulus of elasticity and the Poisson's ratio of the material.

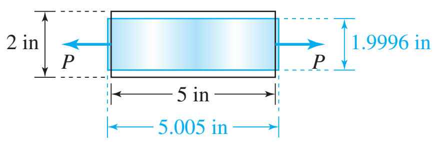


Figure 3.23

3.17 A force $P = 20$ kips is applied to a rigid plate that is attached to a square bar, as shown in Figure P3.24. If the plate moves a distance of 0.005 in., determine the modulus of elasticity.

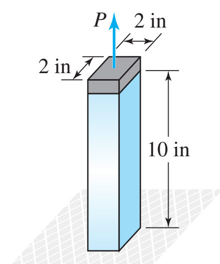


Figure 3.24

3.18 A force $P = 20$ kips is applied to a rigid plate that is attached to a square bar, as shown in Figure P3.25. If the plate moves a distance of 0.0125 in, determine the shear modulus of elasticity. Assume line AB remains straight.

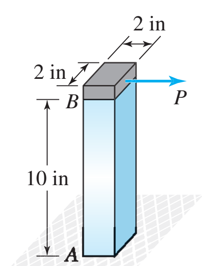


Figure 3.25

3.19 Two rubber blocks of length L and cross section dimension a x b are bonded to rigid plates as shown in Figure P3.19. Point A was observed to move downwards by 0.02 in. when the weight W = 900 lb was hung from the middle plate. Determine the shear modulus of elasticity using small strain approximation. Use L= 12 in., a = 3 in., and b = 2 in.

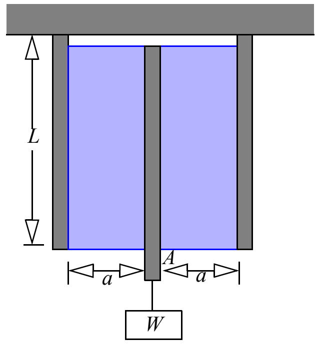


Figure P3.19

3.20 Two rubber blocks with a shear modulus of 1.0 MPa and length L and of cross section dimensions a x b are bonded to rigid plates as shown in Figure P3.19. Using the small-strain approximation, determine the displacement of point A, if a weight of 500 N is hung from the middle plate. Use L = 200 mm, a = 45 mm, and b = 60 mm.

3.21 Two rubber blocks with a shear modulus of 750 psi and length L and cross section dimension a x b are bonded to rigid plates as shown in Figure P3.19. If the allowable shear stress in the rubber is 15 psi, and allowable deflection is 0.03 in., determine the maximum weight W that can be hung from the middle plate using small strain approximation. Use L = 12 in., a = 2 in., and b = 3 in.

3.22 Two rubber blocks with a shear modulus of G, length L and cross section of dimensions a x b are bonded to rigid plates as shown in Figure P3.19. Obtain the shear stress in the rubber block and the displacement of point A in terms of G, L, W, a, and b.

3.23 A circular bar of 200-mm length and 20-mm diameter is subjected to a tension test. Due to an axial force of 77 kN, the bar is seen to elongate by 4.5 mm and the diameter is seen to reduce by 0.162 mm. Determine the modulus of elasticity and the shear modulus of elasticity.

3.24 A circular bar of 6-in. length and 1-in. diameter is made from a material with a modulus of elasticity E = 30,000 ksi and a Poisson's ratio $\nu = \frac{1}{3}$. Determine the change in length and diameter of the bar when a force of 20 kips is applied to the bar.

3.25 A circular bar of 400 mm length and 20 mm diameter is made from a material with a modulus of elasticity E = 180 GPa and a Poisson's ratio ν = 0.32. Due to a force the bar is seen to elongate by 0.5 mm. Determine the change in diameter and the applied force.

3.26 A 25 mm × 25 mm square bar is 500 mm long and is made from a material that has a Poisson's ratio of $\frac{1}{3}$. In a tension test, the bar is seen to elongate by 0.75 mm. Determine the percentage change in volume of the bar.

3.27 A circular bar of 50 in. length and 1 in. diameter is made from a material with a modulus of elasticity E = 28,000 ksi and a Poisson's ratio ν = 0.32. Determine the percentage change in volume of the bar when an axial force of 20 kips is applied.

3.28 An aluminum rectangular bar has a cross section of 25 mm × 50 mm and a length of 500 mm. The modulus of elasticity E = 70 GPa and the Poisson's ratio ν = 0.25. Determine the percentage change in the volume of the bar when an axial force of 300 kN is applied.

3.29 A circular bar of length L and diameter d is made from a material with a modulus of elasticity E and a Poisson's ratio ν. Assuming small strain, show that the percentage change in the volume of the bar when an axial force P is applied and given as $400P(1-2\nu)/(E\pi d^2)$. Note the percentage change is zero when ν = 0.5.

3.30 A rectangular bar has a cross-sectional dimensions $a \times b$ and a length L. The bar material has a modulus of elasticity E and a Poisson's ratio ν. Assuming small strain, show that the percentage change in the volume of the bar when an axial force P is applied given by $100P(1 - 2\nu)/Eab$. Note the percentage change is zero when $\nu = 0.5$.

Strain energy

3.31 What is the strain energy in the bar of Problem 3.16.?

3.32 What is the strain energy in the bar of Problem 3.17?

3.33 What is the strain energy in the bar of Problem 3.18?

3.34 A circular bar of length L and diameter of d is made from a material with a modulus of elasticity E and a Poisson's ratio ν. In terms of the given variables, what is the linear strain energy in the bar when axial load P is applied to the bar?

3.35 A rectangular bar has a cross-sectional dimensions $a \times b$ and a length L. The bar material has a modulus of elasticity E and a Poisson's ratio ν. In terms of the given variables, what is the linear strain energy in the bar when axial load P is applied to the bar?

3.36 For the material having the stress–strain curve shown in Figure P3.3, determine (a) the modulus of resilience (using the proportional limit to approximate the yield point); (b) the strain energy density at a stress level of 420 MPa; (c) the complementary strain energy density at a stress level of 420 MPa; (d) the modulus of toughness.

3.37 For the material having the stress–strain curve shown in Figure P3.6, determine (a) the modulus of resilience (using the proportional limit to approximate the yield point); (b) the strain energy density at a stress level of 72 ksi; (c) the complementary strain energy density at a stress level of 72 ksi; (d) the modulus of toughness.

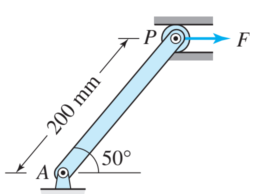
3.38 For the metal alloy given in Problem 3.12, determine (a) the modulus of resilience (using the proportional limit to approximate the yield point); (b) the strain energy density at a stress level of 1400 MPa; (c) the complementary strain energy density at a stress level of 1400 MPa; (d) the modulus of toughness.

3.39 For the mild steel given in Problem 3.13, determine (a) the modulus of resilience (using the proportional limit to approximate the yield point); (b) the strain energy density at a stress level of 50 ksi; (c) the complementary strain energy density at a stress level of 50 ksi; (d) the modulus of toughness.

Logic in mechanics

3.40 The roller at P slides in the slot by an amount $\delta_P = 0.25$ mm due to the force F, as shown in Figure P3.40. Member AP has a cross-sectional area $A = 100\ mm^2$ and a modulus of elasticity $E = 200$ GPa. Determine the force applied F.

Figure P3.40



3.41 The roller at P slides in the slot by an amount $\delta_P = 0.25$ mm due to the force F, as shown in Figure P3.41. Member AP has a cross-sectional area $A = 100\ mm^2$ and a modulus of elasticity $E = 200$ GPa. Determine the applied force F.

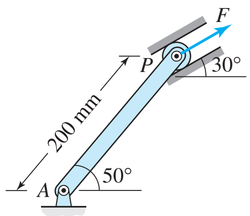


Figure P3.41

3.42 A roller slides in a slot by the amount $\delta_P = 0.01$ in. in the direction of the force F, as shown in Figure P3.42. Each bar has a cross-sectional area $A = 0.2$ in.^{2} and a modulus of elasticity $E = 30{,}000$ ksi. Bars AP and BP have lengths $L_{AP} = 8$ in. and $L_{BP} = 10$ in., respectively. Determine the applied force F.

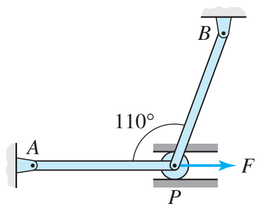


Figure P3.42

3.43 A roller slides in a slot by the amount $\delta_P = 0.25$ mm in the direction of the force F, as shown in Figure P3.43. Each bar has a cross-sectional area $A = 100$ mm^2 and a modulus of elasticity $E = 200$ GPa. Bars AP and BP have lengths $L_{AP} = 200$ mm and $L_{BP} = 250$ mm, respectively. Determine the applied force F.

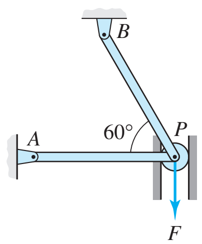


Figure P3.43

3.44 A roller slides in a slot by the amount $\delta_P = 0.25$ mm in the direction of the force F as shown in Figure P3.44. Each bar has a cross-sectional area $A = 100$ mm^2 and a modulus of elasticity $E = 200$ GPa. Bars AP and BP have lengths $L_{AP} = 200$ mm and $L_{BP} = 250$ mm, respectively. Determine the applied force F.

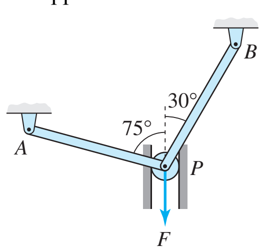


Figure P3.44

3.45 A little boy shoots paper darts at his friends using a rubber band that has an unstretched length of 7 in. The piece of rubber band between points A and B is pulled to form the two sides AC and CB of a triangle, as shown in Figure P3.45. Assume the same normal strain in AC and CB, and the rubber band around the thumb and forefinger is a total of 1 in. The cross-sectional area of the band is $\frac{1}{128}$ in.^{2}, and the rubber has a modulus of elasticity $E = 150$ psi. Determine the approximate force F and the angle θ at which the paper dart leaves the boy's hand.

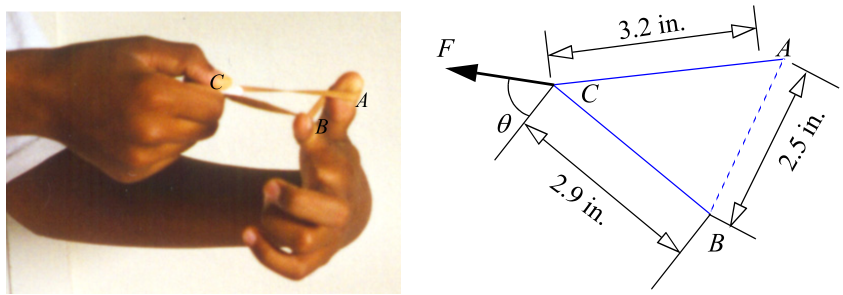


Figure P3.45

3.46 Three poles are pin connected to a ring at P and to the supports on the ground. The coordinates of the four points are given in Figure P3.46. All poles have cross-sectional areas $A = 1$ in.^{2} and a modulus of elasticity $E = 10{,}000$ ksi. If under the action of force F the ring at P moves vertically by the distance $\delta_P = 2$ in., determine the force F.

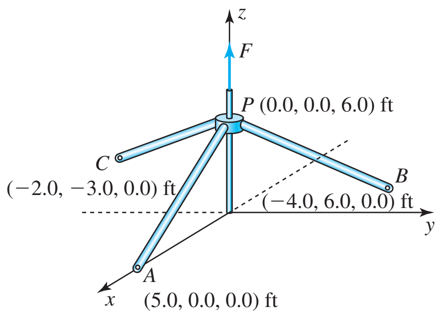


Figure P3.46

3.47 A gap of 0.004 in. exists between a rigid bar and bar A before a force F is applied (Figure P3.47). The rigid bar is hinged at point C. Due to force F the strain in bar A was found to be -500 μin/in. The lengths of bars A and B are 30 in. and 50 in., respectively. Both bars have cross-sectional areas $A = 1$ in.^{2} and a modulus of elasticity $E = 30{,}000$ ksi. Determine the applied force F.

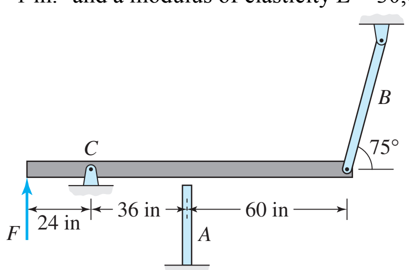


Figure P3.47

3.48 The cable between two poles shown in Figure P3.48 is taut before the two traffic lights are hung on it. The lights are placed symmetrically at 1/3 the distance between the poles. The cable has a diameter of 1/16 in. and a modulus of elasticity of 28,000 ksi. Determine the weight of the traffic lights if the cable sags as shown. Use entire cable to calculate average values.

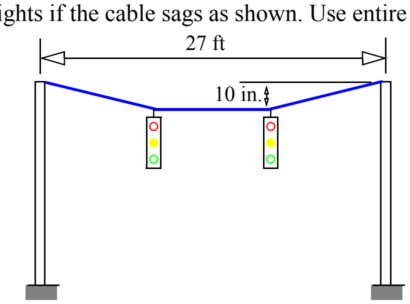


Figure P3.48

3.49 A steel bolt (E_s= 200 GPa) of 25 mm diameter passes through an aluminum (E_{al} = 70 GPa) sleeve of thickness 4 mm and outside diameter of 48 mm as shown in Figure P3.49. Due to the tightening of the nut the rigid washers move towards each other by 0.75 mm. (a) Determine the average normal stress in the sleeve and the bolt. (b) What is the extension of the bolt?

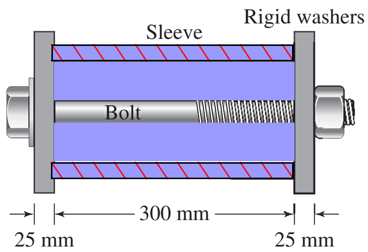


Figure P3.49

3.50 The pins in the truss shown in Figure P3.50 are displaced by *u* and v in the *x* and *y* directions, respectively, as shown in Table P3.50. All rods in the truss have cross-sectional areas A = 100 mm^2 and a modulus of elasticity E = 200 GPa. Determine the external forces P_1 and P_2 in the truss.

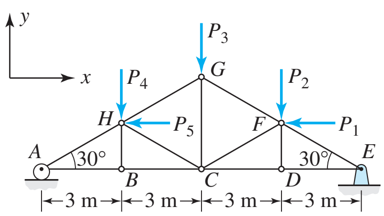


Figure P3.50

TABLE P3.50

u_A = –4.6765 mm	v_A = 0
u_B = –3.3775 mm	v_B = –8.8793 mm
u_C = –2.0785 mm	v_C = –9.7657 mm
u_D = –1.0392 mm	v_D = –8.4118 mm
u_E = 0.0000 mm	v_E = 0.0000 mm
u_F = –3.2600 mm	v_F = –8.4118 mm
u_G = –2.5382 mm	v_G = –9.2461 mm
u_H = –1.5500 mm	v_H = –8.8793 mm

3.51 The pins in the truss shown in Figure P3.50 are displaced by *u* and v in the *x* and *y* directions, respectively, as shown in Table P3.50. All rods in the truss have cross-sectional areas A = 100 mm^2 and a modulus of elasticity E = 200 GPa. Determine the external force P_3 in the truss.

3.52 The pins in the truss shown in Figure P3.50 are displaced by *u* and v in the *x* and *y* directions, respectively, as shown in Table P3.50. All rods in the truss have cross-sectional areas A = 100 mm^2 and a modulus of elasticity E = 200 GPa. Determine the external forces P_4 and P_5 in the truss.

Factor of safety

3.53 A joint in a wooden structure shown in Figure P3.53 is to be designed for a factor of safety of 3. If the average failure stress in shear on the surface *BCD* is 1.5 ksi and the average failure bearing stress on the surface *BEF* is 6 ksi, determine the smallest dimensions *h* and *d* to the nearest $\frac{1}{16}$ in.

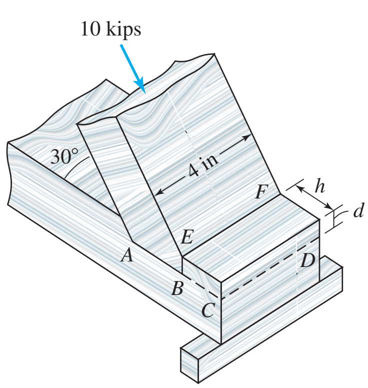


Figure P3.53

3.54 A 125 kg light is hanging from a ceiling by a chain as shown in Figure P3.54. The links of the chain are loops made from a thick wire. Determine the minimum diameter of the wire to the nearest millimeter for a factor of safety of 3. The normal failure stress for the wire is 180 MPa.

Figure P3.54

3.55 A light is hanging from a ceiling by a chain as shown in Figure P3.54. The links of the chain are loops made from a thick wire with a diameter of $\frac{1}{8}$ in. The normal failure stress for the wire is 25 ksi. For a factor of safety of 4, determine the maximum weight of the light to the nearest pound.

3.56 Determine the maximum weight W that can be suspended using cables, as shown in Figure P3.56, for a factor of safety of 1.2. The cable's fracture stress is 200 MPa, and its diameter is 10 mm.

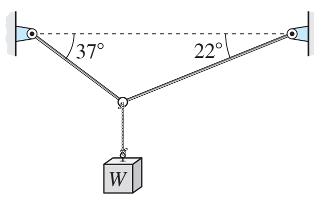


Figure P3.56

3.57 The cable in Figure P3.56 has a fracture stress of 30 ksi and is used for suspending the weight W = 2500 lb. For a factor of safety of 1.25, determine the minimum diameter of the cables to the nearest $\frac{1}{16}$ in. that can be used.

3.58 An adhesively bonded joint in wood is fabricated as shown in Figure P3.58. For a factor of safety of 1.25, determine the minimum overlap length L and dimension h to the nearest $\frac{1}{8}$ in. The shear strength of the adhesive is 400 psi and the wood strength is 6 ksi in tension.

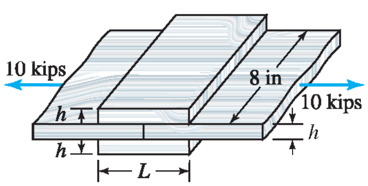


Figure P3.58

3.59 A joint in a truss has the configuration shown in Figure P3.59. Determine the minimum diameter of the pin to the nearest millimeter for a factor of safety of 2.0. The pin's failure stress in shear is 300 MPa.

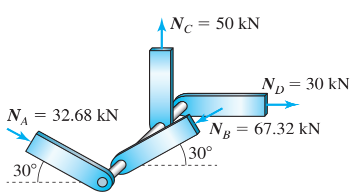


Figure P3.59

3.60 The shear stress on the cross section of the wire of a helical spring (Figure P3.60) is given by $\tau = K(8PC/\pi d^2)$, where P is the force on the spring, d is the diameter of the wire from which the spring is constructed, C is called the spring index, given by the ratio $C = D/d$, D is the diameter of the coiled spring, and K is called the *Wahl factor*, as given below.

$$K = \frac{4C-1}{4C-4} + \frac{0.615}{C}$$

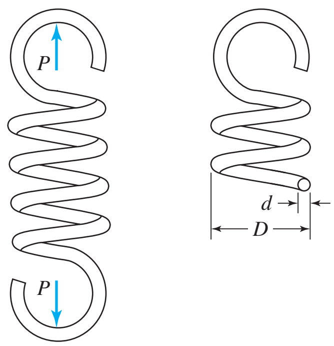


Figure P3.60

The spring is to be designed to resist a maximum force of 1200 N and must have a factor of safety of 1.1 in yield. The shear stress in yield is 350 MPa. Make a table listing admissible values of C and d for 4 mm $\le d \le$ 16 mm in steps of 2 mm.

3.61 Two cast-iron pipes are held together by a steel bolt, as shown in Figure P3.61. The outer diameters of the two pipes are 2 in. and 2 3/4 in., and the wall thickness of each pipe is 1/4 in. The diameter of the bolt is 1/2 in. The yield strength of cast iron is 25 ksi in tension and steel is 15 ksi in shear. What is the maximum force P to the nearest pound this assembly can transmit for a factor of safety of 1.2?

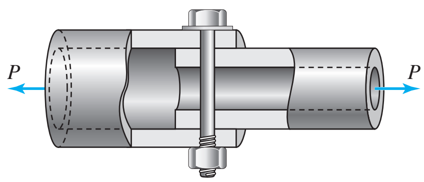


Figure P3.61

3.62 A coupling of diameter 250-mm is assembled using 6 bolts of diameter 12.5 mm as shown in Figure P3.62. The holes for the bolts are drilled with center on a circle of diameter 200 mm. A factor of safety of 1.5 for the assembly is desired. If the shear strength of the bolts is 300 MPa, determine the maximum torque that can be transferred by the coupling.

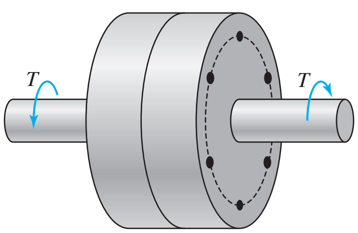


Figure P3.62

Stretch yourself

3.63 A circular rod of 15-mm diameter is acted upon by a distributed force *p(x)* that has the units of kN/m, as shown in Figure P3.63. The modulus of elasticity of the rod is 70 GPa. Determine the distributed force *p(x)* if the displacement *u(x)* in *x* direction is $u(x) = 30(x - x^2)10^{-6}$ m with x is measured in meters.

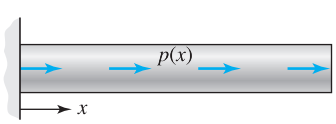


Figure P3.63

3.64 A circular rod of 15-mm diameter is acted upon by a distributed force *p(x)* that has the units of kN/m, as shown in Figure P3.63. The modulus of elasticity of the rod is 70 GPa. Determine the distributed force *p(x)* if the displacement *u(x)* in *x* direction is $u(x) = 50(x^2 - 2x^3)10^{-6}$ m with x is measured in meters.

3.65 Consider the beam shown in Figure P3.65. The displacement in the x direction due to the action of the forces, was found to be $u = [(60x + 80xy - x^2y)/180]10^{-3}$ in. The modulus of elasticity of the beam is 30,000 ksi. Determine the statically equivalent internal normal force $\boldsymbol{N}$ and the internal bending moment $\boldsymbol{M}_z$ acting at point O at a section at $x = 20$ in. Assume an unknown shear stress is acting on the cross-section.

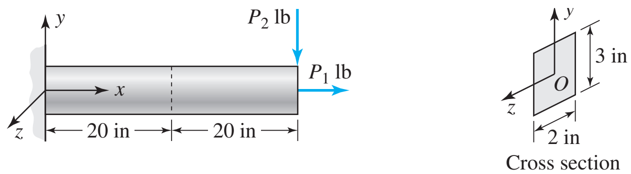


Figure P3.65

Computer problems

3.66 Assume that the stress–strain curve *after* yield stress in Problem 3.12 is described by the quadratic equation $\sigma = a + b\varepsilon + c\varepsilon^2$. (a) Determine the coefficients a, b, and c by the least-squares method. (b) Find the tangent modulus of elasticity at a stress level of 1400 MPa.

3.67 Assume that the stress–strain curve *after* yield stress in Problem 3.13 is described by the quadratic equation $\sigma = a + b\varepsilon + c\varepsilon^2$. (a) Determine the coefficients a, b, and c by the least-squares method. (b) Find the tangent modulus of elasticity at a stress level of 50 ksi.

3.68 Marks were made on the cord used for tying the canoe on top of the car in Example 3.5. These marks were made every 2 in. to produce a total of 20 segments. The stretch cord is symmetric with respect to the top of the canoe. The starting point of the first segment is on the carrier rail of the car and the end point of the tenth segment is on the top of the canoe. The measured length of each segment is as shown in Table 3.68. Determine (a) the tension in the cord of each segment; (b) the force exerted by the cord on the carrier of the car. Use the modulus of elasticity $E = 510$ psi and the diameter of the stretch cord as 1/2 in.

TABLE P3.68

Segment Number	Deformed Length (inches)
1	3.4
2	3.4
3	3.4
4	3.4
5	3.4
6	3.4
7	3.1
8	2.7
9	2.3
10	2.2

3.69 Marks were made on the cord used for tying the canoe on top of the car in Example 3.5. These marks were made every 2 in. to produce a total of 20 segments. The stretch cord is symmetric with respect to the top of the canoe. The starting point of the first segment is on the carrier rail of the car and the end point of the tenth segment is on the top of the canoe. The measured length of each segment is as shown in Table 3.68. Determine (a) the tension in the cord of each segment; (b) the force exerted by the cord on the carrier of the car. Use the diameter of the stretch cord as 1/2 in. and the following equation for the stress–strain curve:

$$\sigma = \begin{cases} 1020\varepsilon - 1020\varepsilon^2 \text{ psi} & \varepsilon < 0.5 \\ 255 \text{ psi} & \varepsilon \geq 0.5 \end{cases}$$

3.70 Marks were made on the cord used for tying the canoe on top of the car in Example 3.5. These marks were made every 2 in. to produce a total of 20 segments. The stretch cord is symmetric with respect to the top of the canoe. The starting point of the first segment is on the carrier rail of the car and the end point of the tenth segment is on the top of the canoe. The measured length of each segment is as shown in Table 3.68. Determine: (a) the tension in the cord of each segment; (b) the force exerted by the cord on the carrier of the car. Use the Poisson's ratio $\nu = \frac{1}{2}$ and the initial diameter of 1/2 in. and calculate the diameter in the deformed position for each segment. Use the stress–strain relationship given in Problem 3.69.

3.4 ISOTROPY AND HOMOGENEITY

The description of a material as isotropic or homogeneous are acquiring greater significance with the development of new materials. In composites (See Section 3.12.3) two or more materials are combined together to produce a stronger or stiffer material. Both material descriptions are approximations influenced by several factors. As will be seen in this section four possible descriptions are: Isotropic–homogeneous; anisotropic–homogeneous; isotropic–nonhomogeneous; and anisotropic–nonhomogeneous.

The number of material constants that need to be measured depends on the material model we want to incorporate into our analysis. Any material model is the relationship between stresses and strains—the simplest model, a linear relationship. With no additional assumptions, the linear relationship of the six strain components to six stress components can be written

$$\begin{aligned}
\varepsilon_{xx} &= C_{11}\sigma_{xx} + C_{12}\sigma_{yy} + C_{13}\sigma_{zz} + C_{14}\tau_{yz} + C_{15}\tau_{zx} + C_{16}\tau_{xy} \\
\varepsilon_{yy} &= C_{21}\sigma_{xx} + C_{22}\sigma_{yy} + C_{23}\sigma_{zz} + C_{24}\tau_{yz} + C_{25}\tau_{zx} + C_{26}\tau_{xy} \\
\varepsilon_{zz} &= C_{31}\sigma_{xx} + C_{32}\sigma_{yy} + C_{33}\sigma_{zz} + C_{34}\tau_{yz} + C_{35}\tau_{zx} + C_{36}\tau_{xy} \\
\gamma_{yz} &= C_{41}\sigma_{xx} + C_{42}\sigma_{yy} + C_{43}\sigma_{zz} + C_{44}\tau_{yz} + C_{45}\tau_{zx} + C_{46}\tau_{xy} \\
\gamma_{zx} &= C_{51}\sigma_{xx} + C_{52}\sigma_{yy} + C_{53}\sigma_{zz} + C_{54}\tau_{yz} + C_{55}\tau_{zx} + C_{56}\tau_{xy} \\
\gamma_{xy} &= C_{61}\sigma_{xx} + C_{62}\sigma_{yy} + C_{63}\sigma_{zz} + C_{64}\tau_{yz} + C_{65}\tau_{zx} + C_{66}\tau_{xy}
\end{aligned} \tag{3.12}$$

Equation (3.12) implies that we need 36 material constants to describe the most general linear relationship between stress and strain. However, it can be shown that the matrix formed by the constants C_{ij} is symmetric (i.e., $C_{ij} = C_{ji}$, where i and j can be any number from 1 to 6). This symmetry is due to the requirement that the strain energy always be positive, but the proof is beyond the scope of this book. The symmetry reduces the maximum number of independent constants to 21 for the most general linear relationship between stress and strain. (Section 3.12.1 describes the controversy over the number of independent constants required in a linear stress–strain relationship.)

Equation (3.12) presupposes that the relation between stress and strain in the x direction is different from the relation in the y or z direction. Alternatively, Equation (3.12) implies that if we apply a force (stress) in the x direction and observe the deformation (strain), then this deformation will differ from the deformation produced if we apply the same force in the y direction. This phenomenon is not observable by the naked eye for most metals, but if we were to look at the metals at the crystal-size level, then the number of constants needed to describe the stress–strain relationship depends on the crystal structure. Thus we need to ask at what level we are conducting the analysis—eye level or crystal size? If we average the impact of the crystal structure at the eye level, then we have defined the simplest material. An **isotropic material** has stress–strain relationships that are independent of the orientation of the coordinate system at a point.

An **anisotropic material** is a material that is not isotropic. The most general anisotropic material requires 21 independent material constants to describe a linear stress–strain relationships. An isotropic body requires only two independent material constants to describe a linear stress–strain relationships (See Example 9.8 and Problem 9.81). Between the isotropic material and the most general anisotropic material lie several types of materials, which are discussed briefly in Section 3.11.2. The degree of difference in material properties with orientation, the scale at which the analysis is being conducted, and the kind of information that is desired from the analysis are some of the factors that influence whether we treat a material as isotropic or anisotropic.

There are many constants used to describe relate stresses and strains (see Problems 3.97 and 3.109), but for isotropic materials only two are independent. That is, all other constants can be found if any two constants are known. The three constants that we shall encounter most in this book are the modulus of elasticity E, the shear modulus of elasticity G, and the Poisson's ratio ν. In Example 9.8 we shall show that for isotropic materials

$$G = \frac{E}{2(1+\nu)} \tag{3.13}$$

Homogeneity is another approximation that is often used to describe a material behavior. A **homogeneous material** has same the material properties at all points in the body. Alternatively, if the material constants C_{ij} are functions of the coordinates x, y, or z, then the material is called nonhomogeneous.

Most materials at the atomic level, the crystalline level, or the grain-size level are nonhomogeneous. The treatment of a material as homogeneous or nonhomogeneous depends once more on the type of information that is to be obtained from the analysis. Homogenization of material properties is a process of averaging different material properties by an overall material property. Any body can be treated as a homogeneous body if the scale at which the analysis is conducted is made sufficiently large.

3.5 GENERALIZED HOOKE'S LAW FOR ISOTROPIC MATERIALS

The equations relating stresses and strains at a *point* in three dimensions are called the **generalized Hooke's law**. The generalized Hooke's law can be developed from the definitions of the three material constants E, ν, and G and the assumption of isotropy. No assumption of homogeneity needs to be made, as the generalized Hooke's law is a stress–strain relationship at a point. In Figure 3.26 normal stresses are applied one at a time. From the definition of the modulus of elasticity we can obtain the strain in the direction of the applied stress, which then is used to get the strains in the perpendicular direction by using the definition of Poisson's ratio. From Figure (3.26*a*), (3.26*b*), and (3.26*c*) we obtain

$$\varepsilon_{xx}^{(1)} = \frac{\sigma_{xx}}{E} \qquad \varepsilon_{yy}^{(1)} = -\nu\varepsilon_{xx}^{(1)} = -\nu\left(\frac{\sigma_{xx}}{E}\right) \qquad \varepsilon_{zz}^{(1)} = -\nu\varepsilon_{xx}^{(1)} = -\nu\left(\frac{\sigma_{xx}}{E}\right)$$

$$\varepsilon_{xx}^{(2)} = -\nu\varepsilon_{yy}^{(2)} = -\nu\left(\frac{\sigma_{yy}}{E}\right) \qquad \varepsilon_{yy}^{(2)} = \frac{\sigma_{yy}}{E} \qquad \varepsilon_{zz}^{(2)} = -\nu\varepsilon_{yy}^{(2)} = -\nu\left(\frac{\sigma_{yy}}{E}\right)$$

$$\varepsilon_{xx}^{(3)} = -\nu\varepsilon_{zz}^{(3)} = -\nu\left(\frac{\sigma_{zz}}{E}\right) \qquad \varepsilon_{yy}^{(3)} = -\nu\varepsilon_{zz}^{(3)} = -\nu\left(\frac{\sigma_{zz}}{E}\right) \qquad \varepsilon_{zz}^{(3)} = \frac{\sigma_{zz}}{E}$$

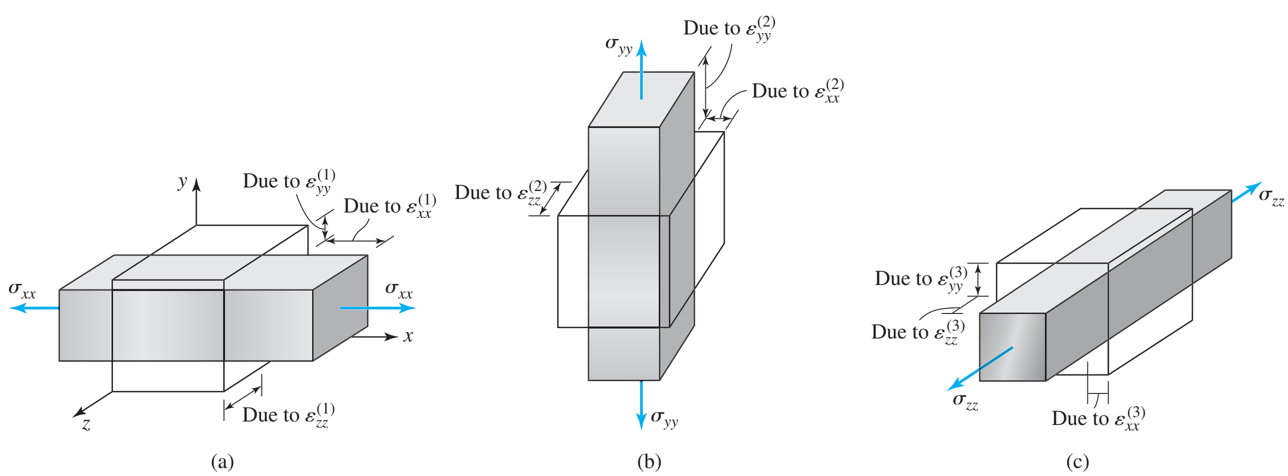


Figure 3.26 Derivation of the generalized Hooke's law.

The use of the same E and ν to relate stresses and strains in different directions implicitly assumes isotropy. Notice that no change occurs in the right angles from the application of normal stresses. Thus no shear strain is produced due to normal stresses in a fixed coordinate system for an isotropic material.

Assuming the material is linearly elastic, we can use the principle of superposition to obtain the total strain $\varepsilon_{ii} = \varepsilon_{ii}^{(1)} + \varepsilon_{ii}^{(2)} + \varepsilon_{ii}^{(3)}$, as shown in Equations (3.14a) through (3.14c). From the definition of shear modulus given in Equation (3.3), we obtain Equations (3.14d) through (3.14f).

Generalized Hooke's law:

$$\varepsilon_{xx} = \frac{\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})}{E} \tag{3.14a}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})}{E} \tag{3.14b}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})}{E} \tag{3.14c}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{3.14d}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \tag{3.14e}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \tag{3.14f}$$

The equations are valid for nonhomogeneous material. The nonhomogeneity will make the material constants E, ν, and G functions of the spatial coordinates. The use of Poisson's ratio to relate strains in perpendicular directions is valid not only for Cartesian coordinates but for any orthogonal coordinate system. Thus the generalized Hooke's law may be written for *any orthogonal* coordinate system, such as spherical and polar coordinate systems.

An alternative form[1] for Equations (3.14a) through (3.14c), which may be easier to remember, is the matrix form

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} \tag{3.15}$$

3.6 PLANE STRESS AND PLANE STRAIN

In Chapters 1 and 2 two-dimensional problems of plane stress and plane strain, respectively. Taking the two definitions and using Equations (3.14a), (3.14b), (3.14c), and (3.14f), we obtain the matrices shown in Figure 3.27. The difference between the two idealizations of material behavior is in the zero and nonzero values of the normal strain and normal stress in the z direction. In plane stress $\sigma_{zz} = 0$, which from Equation (3.14c) implies that the normal strain in the z direction is $\varepsilon_{zz} = -\nu(\sigma_{xx} + \sigma_{yy})/E$. In plane strain $\varepsilon_{zz} = 0$, which from Equation (3.14c) implies that the normal stress in the z direction is $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$. .

Plane stress $\longrightarrow \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Generalized Hooke's law}} \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \end{bmatrix}$

Plane strain $\longrightarrow \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Generalized Hooke's law}} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \end{bmatrix}$

Figure 3.27 Stress and strain matrices in plane stress and plane strain.

Figure 3.28 shows two plates on which only compressive normal stresses in the x and y directions are applied. The top and bottom surfaces on the plate in Figure 3.28*a* are free surfaces (plane stress), but because the plate is free to expand, the deformation (strain) in the z direction is not zero. The plate in Figure 3.28*b* is constrained from expanding in the z direction by

[1]Another alternative is $\varepsilon_{ii} = [(1+\nu)\sigma_{ii} - \nu I_1]/E$, where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$.

the rigid surfaces. As the material pushes on the plate, a reaction force develops, and this reaction force results in a nonzero value of normal stress in the *z* direction. Plane stress or plane strain are often approximations to simplify analysis. Plane stress approximation is often made for thin bodies, such as the metal skin of an aircraft. Plane strain approximation is often made for thick bodies, such as the hull of a submarine.

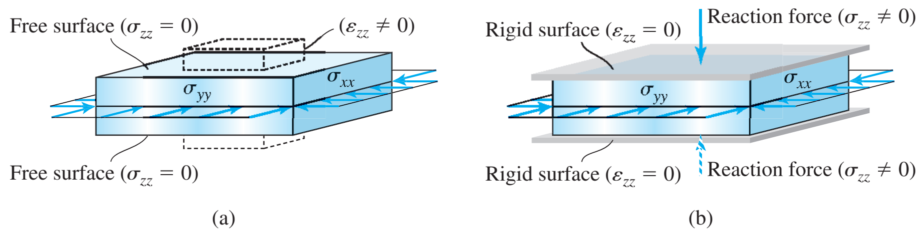


Figure 3.28 (*a*) Plane stress. (*b*) Plane strain.

It should be recognized that in plane strain and plane stress conditions there are only three independent quantities, even though the nonzero quantities number more than 3. For example, if we know σ_{xx}, σ_{yy}, and τ_{xy}, then we can calculate ε_{xx}, ε_{yy}, γ_{xy}, ε_{zz}, and σ_{zz} for plane stress and plane strain. Similarly, if we know ε_{xx}, ε_{yy}, and γ_{xy}, then we can calculate σ_{xx}, σ_{yy}, τ_{xy}, σ_{zz}, and ε_{zz} for plane stress and plane strain. Thus in both plane stress and plane strain the number of independent stress or strain components is 3, although the number of nonzero components is greater than 3. Examples 3.7 and 3.8 elaborate on the difference between plane stress and plane strain conditions and the difference between nonzero and independent quantities.

EXAMPLE 3.7

The stresses at a point on steel were found to be σ_{xx} = 15 ksi (T), σ_{yy} = 30 ksi (C), and τ_{xy} = 25 ksi. Using E = 30,000 ksi and G = 12,000 ksi, determine the strains ε_{xx}, ε_{yy}, γ_{xy}, ε_{zz} and the stress σ_{zz} assuming (a) the point is in a state of plane stress. (b) the point is in a state of plane strain.

PLAN

In both cases the shear strain is the same and can be calculated using Equation (3.14d). (a) For plane stress $\sigma_{zz} = 0$ and the strains ε_{xx}, ε_{yy}, and ε_{zz} can be found from Equations (3.14a), (3.14b), and (3.14c), respectively. (b) For plane strain $\varepsilon_{zz} = 0$ and Equation (3.14c) can be used to find σ_{zz}. The stresses σ_{xx}, σ_{yy}, and σ_{zz} can be substituted into Equations (3.14a) and (3.14b) to calculate the normal strains ε_{xx} and ε_{yy}.

SOLUTION

From Equation (3.14d)

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{25 \text{ ksi}}{12,000 \text{ ksi}} = 0.002083 \quad \text{(E1)}$$

ANS. $\gamma_{xy} = 2083\ \mu$

The Poisson's ratio can be found from Equation (3.13),

$$G = \frac{E}{2(1+\nu)} \quad \text{or} \quad 12{,}000 \text{ ksi} = \frac{30{,}000 \text{ ksi}}{2(1+\nu)} \quad \text{or} \quad \nu = 0.25 \quad \text{(E2)}$$

(a) *Plane stress*: The normal strains in the *x*, *y*, and *z* directions are found from Equations (3.14a), (3.14b), and (3.14c), respectively.

$$\varepsilon_{xx} = \frac{\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})}{E} = \frac{15 \text{ ksi} - 0.25(-30 \text{ ksi})}{30{,}000 \text{ ksi}} = 750(10^{-6}) \quad \text{(E3)}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})}{E} = \frac{-30 \text{ ksi} - 0.25(15 \text{ ksi})}{30{,}000 \text{ ksi}} = -1125(10^{-6}) \quad \text{(E4)}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})}{E} = \frac{0 - 0.25(15 \text{ ksi} - 30 \text{ ksi})}{30{,}000 \text{ ksi}} = 125(10^{-6}) \quad \text{(E5)}$$

ANS. $\varepsilon_{xx} = 750\ \mu$ $\quad \varepsilon_{yy} = -1125\ \mu$ $\quad \varepsilon_{zz} = 125\ \mu$

(b) *Plane strain:* From Equation (3.14c), we have

$$\varepsilon_{zz} = \frac{[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]}{E} = 0 \quad \text{or} \quad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = 0.25(15 \text{ ksi} - 30 \text{ ksi}) = -3.75\text{ksi} \tag{E6}$$

ANS. $\sigma_{zz} = 3.75 \text{ ksi (C)}$

The normal strains in the *x* and *y* directions are found from Equations (3.14a) and (3.14b),

$$\varepsilon_{xx} = \frac{\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})}{E} = \frac{15 \text{ ksi} - 0.25(-30 \text{ ksi} - 3.75 \text{ ksi})}{30{,}000 \text{ ksi}} = 781.2(10^{-6}) \tag{E7}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})}{E} = \frac{-30 \text{ ksi} - 0.25(15 \text{ ksi} - 3.75 \text{ ksi})}{30{,}000 \text{ ksi}} = -1094(10^{-6}) \tag{E8}$$

ANS. $\varepsilon_{xx} = 781.2\ \mu \qquad \varepsilon_{yy} = -1094\ \mu$

COMMENTS

1. The three independent quantities in this problem were σ_{xx}, σ_{yy}, and τ_{xy}. Knowing these we were able to find all the strains in plane stress and plane strain.
2. The difference in the values of the strains came from the zero value of σ_{zz} in plane stress and a value of -3.75 ksi in plane strain.

EXAMPLE 3.8

The strains at a point on aluminum ($E = 70$ GPa, $G = 28$ GPa, and $\nu = 0.25$) were found to be $\varepsilon_{xx} = 650\ \mu$, $\varepsilon_{yy} = 300\ \mu$, and $\gamma_{xy} = 750\ \mu$. . Determine the stresses σ_{xx}, σ_{yy}, and τ_{xy} and the strain ε_{zz} assuming the point is in plane stress.

PLAN

The shear strain can be calculated using Equation (3.14d). If we note that $\sigma_{zz} = 0$ and the strains ε_{xx} and ε_{yy} are given, the stresses σ_{xx} and σ_{yy} can be found by solving Equations (3.14a) and (3.14b) simultaneously. The strain ε_{zz} can then be found from Equation (3.14c).

SOLUTION

From Equation (3.14d),

$$\tau_{xy} = G\gamma_{xy} = (28 \times 10^9 \text{N/m}^2)(750 \times 10^{-6}) = 21(10^6) \text{ N/m}^2 \tag{E1}$$

ANS. $\tau_{xy} = 21 \text{ MPa}$

Equations (3.14a) and (3.14b) can be rewritten with $\sigma_{zz} = 0$,

$$\sigma_{xx} - \nu\sigma_{yy} = E\varepsilon_{xx} = (70 \times 10^9 \text{ N/m}^2)(650 \times 10^{-6}) \quad \text{or} \tag{E2}$$

$$\sigma_{xx} - \nu\sigma_{yy} = 45.5 \text{ MPa} \tag{E3}$$

$$\sigma_{yy} - \nu\sigma_{xx} = E\varepsilon_{yy} = (70 \times 10^9 \text{ N/m}^2)(300 \times 10^{-6}) \quad \text{or} \tag{E4}$$

$$\sigma_{yy} - 0.25\sigma_{xx} = 21 \text{ MPa} \tag{E5}$$

Solving Equations (E3) and (E5) we obtain σ_{xx} and σ_{yy}.

ANS. $\sigma_{xx} = 54.1 \text{ MPa (T)} \qquad \sigma_{yy} = 34.5 \text{ MPa (T)}$

From Equation (3.14c) we obtain

$$\varepsilon_{zz} = \frac{\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})}{E} = \frac{0 - 0.25(54.13 + 34.53)10^6}{70 \times 10^9} = -317(10^{-6}) \tag{E6}$$

ANS. $\varepsilon_{zz} = -317\ \mu$

COMMENTS

1. Equations (E3) and (E5) have a very distinct structure. If we multiply either equation by ν and add the product to the other equation, the result will be to eliminate one of the unknowns. Equation (3.17) in Problem 3.104 is developed in this manner and can be used for solving this problem. But this would imply remembering one more formula. We can avoid this by remembering the defined structure of Hooke's law, which is applicable to all types of problems and not just plane stress.
2. Equation (3.18) in Problem 3.105 gives $\varepsilon_{zz} = -[\nu/(1-\nu)](\varepsilon_{xx} + \varepsilon_{yy})$. Substituting $\nu = 0.25$ and $\varepsilon_{xx} = 650\ \mu$, $\varepsilon_{yy} = 300\ \mu$, we obtain $\varepsilon_{zz} = -(0.25/0.75)(650 + 300) = 316.7\ \mu$, as before. This formula is useful if we do not need to calculate stresses, and we will use it in Chapter 9.

QUICK TEST 3.2

Time: 15 minutes/Total: 20 points

Grade yourself using the answers given in Appendix E. Each question is worth two points.

1. What is the difference between an isotropic and a homogeneous material?
2. What is the number of independent material constants needed in a linear stress–strain relationship for an isotropic material?
3. What is the number of independent material constants needed in a linear stress–strain relationship for the most general anisotropic materials?
4. What is the number of independent *stress* components in plane stress problems?
5. What is the number of independent *strain* components in plane stress problems?
6. How many nonzero *strain* components are there in plane stress problems?
7. What is the number of independent *strain* components in plane strain problems?
8. What is the number of independent *stress* components in plane strain problems?
9. How many nonzero *stress* components are there in plane strain problems?
10. Is the value of E always greater than G, less than G, or does it depend on the material? Justify your answer.

PROBLEM SET 3.2

3.71 Write the generalized Hooke's law for isotropic material in cylindrical coordinates (r, θ, z).

3.72 Write the generalized Hooke's law for isotropic material in spherical coordinates (r, θ, ϕ).

In problems 3.73 through 3.78 two material constants and the stress components in the x, y plane are given. Calculate ε_{xx}, ε_{yy}, γ_{xy}, ε_{zz}, *and* σ_{zz} *(a) assuming plane stress; (b) assuming plane strain.*

3.73	E = 200 GPa	$\nu = 0.32$	σ_{xx} = 100 MPa (T)	σ_{yy} = 150 MPa (T)	τ_{xy} = –125 MPa
3.74	E = 70 GPa	G = 28 GPa	σ_{xx} = 225 MPa (C)	σ_{yy} = 125 MPa (T)	τ_{xy} = 150 MPa
3.75	E = 30,000 ksi	$\nu = 0.3$	σ_{xx} = 22 ksi (C)	σ_{yy} = 25 ksi (C)	τ_{xy} = –15 ksi
3.76	E = 10,000 ksi	G = 3900 ksi	σ_{xx} = 15 ksi (T)	σ_{yy} = 12 ksi (C)	τ_{xy} = –10 ksi
3.77	G = 15 GPa	$\nu = 0.2$	σ_{xx} = 300 MPa (C)	σ_{yy} = 300 MPa (T)	τ_{xy} = 150 MPa
3.78	E = 2000 psi	G = 800 psi	σ_{xx} = 100 psi (C)	σ_{yy} = 150psi (T)	τ_{xy} = 100 psi

In problems 3.79 through 3.84 two material constants and the strain components in the x, y plane are given. Calculate σ_{xx}, σ_{yy}, τ_{xy}, σ_{zz}, *and* ε_{zz} *assuming the point is in plane stress.*

3.79	E = 200 GPa	$\nu = 0.32$	$\varepsilon_{xx} = 500\ \mu$	$\varepsilon_{yy} = 400\ \mu$	$\gamma_{xy} = -300\ \mu$
3.80	E = 70 GPa	G = 28 GPa	$\varepsilon_{xx} = 2000\ \mu$	$\varepsilon_{yy} = -1000\ \mu$	$\gamma_{xy} = 1500\mu$
3.81	E = 30,000 ksi	$\nu = 0.3$	$\varepsilon_{xx} = -800\ \mu$	$\varepsilon_{yy} = -1000\ \mu$	$\gamma_{xy} = -500\ \mu$
3.82	E = 10,000 ksi	G = 3900 ksi	$\varepsilon_{xx} = 1500\ \mu$	$\varepsilon_{yy} = -1200\ \mu$)	$\gamma_{xy} = -1000\ \mu$
3.83	G = 15 GPa	$\nu = 0.2$	$\varepsilon_{xx} = -2000\ \mu$	$\varepsilon_{yy} = 2000\ \mu$	$\gamma_{xy} = 1200\ \mu$
3.84	E = 2000 psi	G = 800 psi	$\varepsilon_{xx} = 50\ \mu$	$\varepsilon_{yy} = 75\ \mu$	$\gamma_{xy} = -25\ \mu$

3.85 The cross section of the wooden piece that is visible in Figure P3.85 is 40 mm × 25 mm. The clamped length of the wooden piece in the vice is 125 mm. The modulus of elasticity of wood is $E = 14$ GPa and the Poisson's ratio $\nu = 0.3$. The jaws of the vice exert a uniform pressure of 3.2 MPa on the wood. Determine the average change of length of the wood.

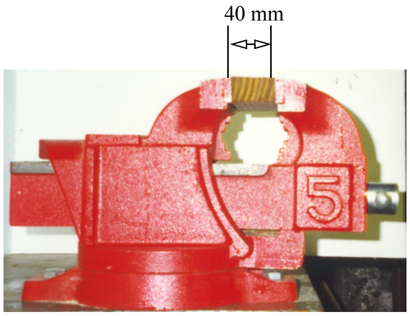


Figure P3.85

3.86 A thin plate ($E = 30{,}000$ ksi, $\nu = 0.25$) under the action of uniform forces deforms to the shaded position, as shown in Figure P3.86. Assuming plane stress, determine the average normal stresses in the x and y directions.

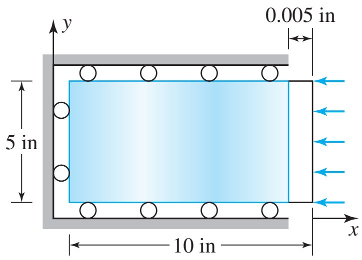


Figure P3.86

3.87 A thin plate (E=30,000 ksi, ν = 0.25) is subjected to a uniform stress σ = 10 ksi as shown in Figure P3.87. Assuming plane stress, determine (a) the average normal stress in y direction; (b) the contraction of the plate in x direction.

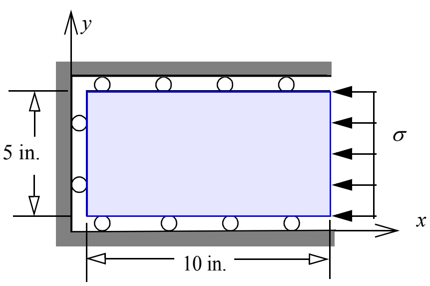


Figure P3.87

3.88 A rubber (E_R=300 psi and ν_R = 0.5) rod of diameter d_R =4 in. is placed in a steel (rigid) tube d_S =4.1 in. as shown in Figure P3.88. What is the smallest value of P that can be applied so that the space between the rubber rod and the steel tube would close.

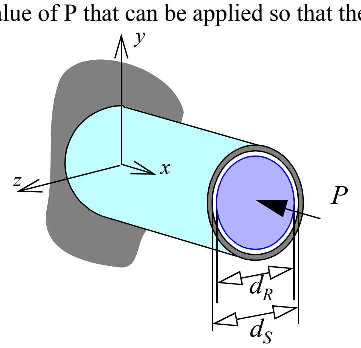


Figure P3.88

3.89 A rubber (E_R= 2.1GPa and ν_R = 0.5) rod of diameter d_R = 200 mm is placed in a steel (rigid) tube d_S = 204 mm as shown in Figure P3.88. If the applied force is P = 10 kN, determine the average normal stress in the y and z direction.

3.90 A 2 in. × 2 in. square with a circle inscribed is stressed as shown Figure P3.90. The plate material has a modulus of elasticity E = 10,000 ksi and a Poisson's ratio ν = 0.25. Assuming plane stress, determine the major and minor axes of the ellipse formed due to deformation.

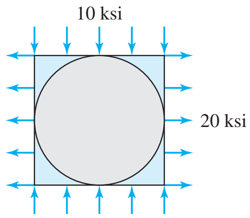


Figure P3.90

3.91 A 2 in. × 2 in. square with a circle inscribed is stressed as shown Figure P3.91. The plate material has a modulus of elasticity $E = 10{,}000$ ksi and a Poisson's ratio $\nu = 0.25$. Assuming plane stress, determine the major and minor axes of the ellipse formed due to deformation.

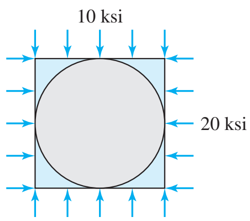


Figure P3.91

3.92 A 50 mm × 50 mm square with a circle inscribed is stressed as shown Figure P3.92. The plate material has a modulus of elasticity $E = 70$ GPa and a Poisson's ratio $\nu = 0.25$. Assuming plane stress, determine the major and minor axes of the ellipse formed due to deformation.

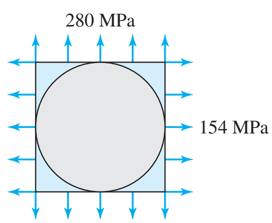


Figure P3.92

3.93 A rectangle inscribed on an aluminum (10,000 ksi, $\nu = 0.25$) plate is observed to deform into the colored shape shown in Figure P3.93. Determine the average stress components σ_{xx}, σ_{yy}, and τ_{xy}.

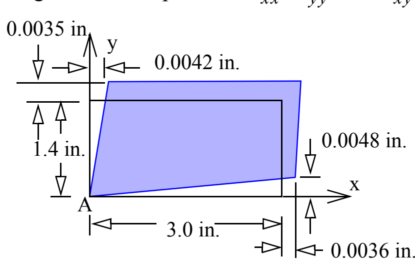


Figure P3.93

3.94 A rectangle inscribed on an steel (E = 210 GPa, $\nu = 0.28$) plate is observed to deform into the colored shape shown in Figure P3.94. Determine the average stress components σ_{xx}, σ_{yy}, and τ_{xy}.

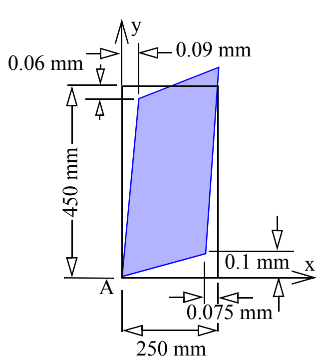


Figure P3.94

3.95 A 5 ft mean diameter spherical steel (E = 30,000 ksi, ν = 0.28) tank has a wall thickness of 3/4 in. Determine the increase in the mean diameter when the gas pressure inside the tank is 600 psi.

3.96 A steel (E = 210 GPa ν = 0.28) cylinder of mean diameter of 1 m and wall thickness of 10 mm has gas at 250 kPa. Determine the increase in the mean diameter due to gas pressure.

3.97 Derive the following relations of normal stresses in terms of normal strain from the generalized Hooke's law:

$$\sigma_{xx} = [(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}]\frac{E}{(1-2\nu)(1+\nu)}$$
$$\sigma_{yy} = [(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz} + \nu\varepsilon_{xx}]\frac{E}{(1-2\nu)(1+\nu)} \qquad (3.16)$$
$$\sigma_{zz} = [(1-\nu)\varepsilon_{zz} + \nu\varepsilon_{xx} + \nu\varepsilon_{yy}]\frac{E}{(1-2\nu)(1+\nu)}$$

An alternative form that is easier to remember is $\sigma_{ii} = 2G\varepsilon_{ii} + \lambda(I_1)$, where i can be x, y, or z; $I_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$; G is the shear modulus; and $\lambda = 2G\nu/(1-2\nu)$ is called *Lame's constant*, after G. Lame (1795–1870).

3.98 For a point in plane stress show that

$$\sigma_{xx} = (\varepsilon_{xx} + \nu\varepsilon_{yy})\frac{E}{1-\nu^2} \qquad \sigma_{yy} = (\varepsilon_{yy} + \nu\varepsilon_{xx})\frac{E}{1-\nu^2} \qquad (3.17)$$

3.99 For a point in plane stress show that

$$\varepsilon_{zz} = -\left(\frac{\nu}{1-\nu}\right)(\varepsilon_{xx} + \varepsilon_{yy}) \qquad (3.18)$$

3.100 Using Equations (3.17) and (3.18), solve for σ_{xx}, σ_{yy}, and ε_{zz} in Problem 3.79.

3.101 Using Equations (3.17) and (3.18), solve for σ_{xx}, σ_{yy}, and ε_{zz} in Problem 3.80.

3.102 Using Equations (3.17) and (3.18), solve for σ_{xx}, σ_{yy}, and ε_{zz} in Problem 3.81.

3.103 Using Equations (3.17) and (3.18), solve for σ_{xx}, σ_{yy}, and ε_{zz} in Problem 3.82.

3.104 Using Equations (3.17) and (3.18), solve for σ_{xx}, σ_{yy}, and ε_{zz} in Problem 3.83.

3.105 Using Equations (3.17) and (3.18), solve for σ_{xx}, σ_{yy}, and ε_{zz} in Problem 3.84.

3.106 For a point in plane strain show that

$$\varepsilon_{xx} = [(1-\nu)\sigma_{xx} - \nu\sigma_{yy}]\frac{1+\nu}{E} \qquad \varepsilon_{yy} = [(1-\nu)\sigma_{yy} - \nu\sigma_{xx}]\frac{1+\nu}{E} \qquad (3.19)$$

3.107 For a point in plane strain show that

$$\sigma_{xx} = [(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}]\frac{E}{(1-2\nu)(1+\nu)} \qquad \sigma_{yy} = [(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{xx}]\frac{E}{(1-2\nu)(1+\nu)} \qquad (3.20)$$

3.108 A differential element subjected to only normal strains is shown in Figure P3.108. The ratio of change in a volume ΔV to the original volume V is called the volumetric strain ε_V, or *dilation*.

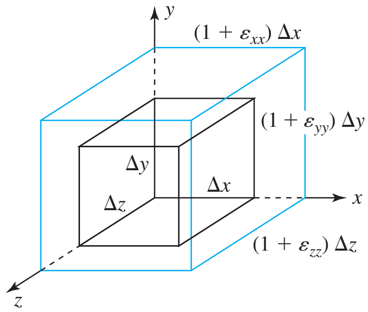


Figure P3.108

For small strain prove

$$\varepsilon_V = \frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \qquad (3.21)$$

3.109 Prove

$$p = -K\varepsilon_V \qquad p = -\left(\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}\right) \qquad K = \frac{E}{3(1-2\nu)} \qquad (3.22)$$

where K is the *bulk modulus* and p is the *hydrostatic pressure* because at a point in a fluid the normal stresses in all directions are equal to $-p$. Note that at $\nu = \frac{1}{2}$ there is no change in volume, regardless of the value of the stresses. Such materials are called *incompressible materials*.

Stretch yourself

An orthotropic material (Section 3.12.3) has the following stress–strain relationship at a point in plane stress:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_x} - \frac{\nu_{yx}}{E_y}\sigma_{yy} \qquad \varepsilon_{yy} = \frac{\sigma_{yy}}{E_y} - \frac{\nu_{xy}}{E_x}\sigma_{xx} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \qquad \frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x} \qquad (3.23)$$

Use Equations (3.23) to solve Problems 3.110 through 3.117 .
The stresses at a point on a free surface of an orthotropic material are given in Problems 3.110 through 3.113 . Also given are the material constants. Using Equations (3.23) solve for the strains ε_{xx}, ε_{yy}, and γ_{xy}.

Problem	σ_{xx}	σ_{yy}	τ_{xy}	E_x	E_y	ν_{xy}	G_{xy}
3.110	5 ksi (C)	8 ksi (T)	6 ksi	7500 ksi	2500 ksi	0.3	1250 ksi
3.111	25 ksi (C)	5 ksi (C)	–8 ksi	25,000 ksi	2000 ksi	0.32	1500 ksi
3.112	200 MPa (C)	80 MPa (C)	–54 MPa	53 GPa	18 GPa	0.25	9 GPa
3.113	300 MPa (T)	50 MPa (T)	60 MPa	180 GPa	15 GPa	0.28	11 GPa

The strains at a point on a free surface of an orthotropic material are given in Problems 3.114 through 3.117 . Also given are the material constants. Using Equation (3.23) solve for the stresses σ_{xx}, σ_{yy}, and τ_{xy}.

Problem	ε_{xx}	ε_{yy}	γ_{xy}	E_x	E_y	ν_{xy}	G_{xy}
3.114	$-1000\ \mu$	$500\ \mu$	$-250\ \mu$	7500 ksi	2500 ksi	0.3	1250 ksi
3.115	$-750\ \mu$	$-250\ \mu$	$400\ \mu$	25,000 ksi	2000 ksi	0.32	1500 ksi
3.116	$1500\ \mu$	$800\ \mu$	$600\ \mu$	53 GPa	18 GPa	0.25	9 GPa
3.117	$1500\ \mu$	$-750\ \mu$	$-450\ \mu$	180 GPa	15 GPa	0.28	11 GPa

3.118 Using Equation (3.23), show that on a free surface of an orthotropic material

$$\sigma_{xx} = \frac{E_x(\varepsilon_{xx} + \nu_{yx}\varepsilon_{yy})}{1 - \nu_{yx}\nu_{xy}} \qquad \sigma_{yy} = \frac{E_y(\varepsilon_{yy} + \nu_{xy}\varepsilon_{xx})}{1 - \nu_{yx}\nu_{xy}} \qquad (3.24)$$

3.7* STRESS CONCENTRATION

Large stress gradients in a small region are called **stress concentration**. These large gradients could be due to sudden changes in geometry, material properties, or loading. We can use our theoretical models to calculate stress away from the regions of large stress concentration according to Saint-Venant's principle, which will be discussed in the next section. These stress values predicted by the theoretical models away from regions of stress concentration are called **nominal stresses**. Figure 3.29 shows photoelastic pictures (see Section 8.4.1) of two structural members under uniaxial tension. Large stress gradients near the circular cutout boundaries cause fringes to be formed. Each color boundary represents a fringe order that can be used in the calculation of the stresses.

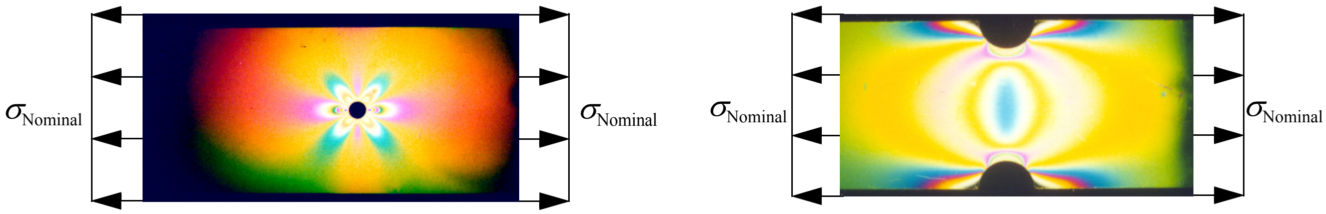


Figure 3.29 Photoelastic pictures showing stress concentration. (Courtesy Professor I. Miskioglu.)

Stress concentration factor is an engineering concept that permits us to extrapolate the results of our elementary theory into the region of large stress concentration where the assumptions on which the theory is based are violated. The stress concentration factor K_{conc} is defined as

$$K_{conc} = \frac{\text{maximum stress}}{\text{nominal stress}} \tag{3.25}$$

The stress concentration factor K_{conc} is found from charts, tables, or formulas that have been determined experimentally, numerically, analytically, or from a combination of the three. Section C.4 in Appendix shows several graphs that can be used in the calculation of stress concentration factors for problems in this book. Additional graphs can be found in handbooks describing different situations. Knowing the nominal stress and the stress concentration factor, the maximum stress can be estimated and used in design or to estimate the factor of safety. Example 3.9 demonstrates the use of the stress concentration factor.

3.8* SAINT-VENANT'S PRINCIPLE

Theories in mechanics of materials are constructed by making assumptions regarding load, geometry, and material variations. These assumptions are usually not valid near concentrated forces or moments, near supports, near corners or holes, near interfaces of two materials, and in flaws such as cracks. Fortunately, however, disturbance in the stress and displacement fields dissipates rapidly as one moves away from the regions where the assumptions of the theory are violated. Saint-Venant's principle states

> Two statically equivalent load systems produce nearly the same stress in regions at a distance that is at least equal to the largest dimension in the loaded region.

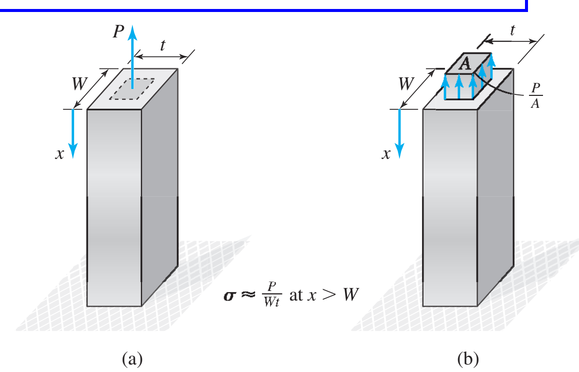


Figure 3.30 Stress due to two statically equivalent load systems.

Consider the two statically equivalent load systems shown in Figure 3.30. By Saint-Venant's principle the stress at a distance W away from the loads will be nearly uniform. In the region at a distance less than W the stress distribution will be different, and it is possible that there are also shear stress components present. In a similar manner, changes in geometry and materials have local effects that can be ignored at distances. We have considered the effect of changes in geometry and an engineering solution to the problem in Section 3.7 on stress concentration.

The importance of Saint-Venant's principle is that we can develop our theories with reasonable confidence away from the regions of stress concentration. These theories provide us with formulas for the calculation of nominal stress. We can then use the stress concentration factor to obtain maximum stress in regions of stress concentration where our theories are not valid.

EXAMPLE 3.9

Finite-element analysis (see Section 4.8) shows that a long structural component in Figure 3.31 carries a uniform axial stress of $\sigma_{nominal}$ = 35 MPa (T). A hole in the center needs to be drilled for passing cables through the structural component. The yield stress of the material is σ_{yield} = 200 MPa. If failure due to yielding is to be avoided, determine the maximum diameter of the hole that can be drilled using a factor of safety of K_{safety} =1.6.

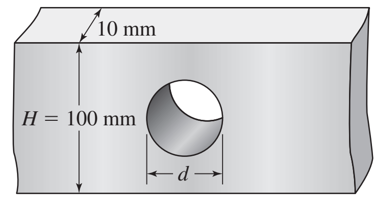


Figure 3.31 Component geometry in Example 3.9.

PLAN

We can compute the allowable (maximum) stress for factor of safety of 1.6 from Equation (3.10). From Equation (3.25) we can find the permissible stress concentration factor. From the plot of K_{gross} in Figure A.13 of Appendix C we can estimate the ratio of d/H. Knowing that H = 100 mm, we can find the maximum diameter d of the hole.

SOLUTION

From Equation (3.10) we obtain the allowable stress:

$$\sigma_{allow} = \frac{\sigma_{yield}}{K_{safety}} = \frac{200 \text{ MPa}}{1.6} = 125 \text{ MPa} \tag{E1}$$

From Equation (3.25) we calculate the permissible stress concentration factor:

$$K_{conc} \le \frac{\sigma_{allow}}{\sigma_{nominal}} = \frac{125 \text{ MPa}}{35 \text{ MPa}} = 3.57 \tag{E2}$$

From Figure A.13 of Appendix C we estimate the ratio of d/H as 0.367. Substituting H = 100 mm we obtain

$$\frac{d}{100 \text{ mm}} \le 0.367 \qquad \text{or} \qquad d \le 36.7 \text{ mm} \tag{E3}$$

For the maximum permissible diameter to the nearest millimeter we round downward.

ANS. $d_{max} = 36 \text{ mm}$

COMMENTS

1. The value of d/H = 0.367 was found from linear interpolation between the value of d/H = 0.34, where the stress concentration factor is 0.35, and the value of d/H = 0.4, where the stress concentration factor is 0.375. These points were used as they are easily read from the graph. Because we are rounding downward in Equation (E3), any value between 0.36 and 0.37 is acceptable. In other words, the third place of the decimal value is immaterial.
2. As we used the maximum diameter of 36 mm instead of 36.7 mm, the effective factor of safety will be slightly higher than the specified value of 1.6, which makes this design a conservative design.
3. Creating the hole will change the stress around its. By per Saint-Venant's principle, the stress field far from the hole will not be significantly affected. This justifies the use of nominal stress without the hole in our calculation.

3.9* THE EFFECT OF TEMPERATURE

A material expands with an increase in temperature and contracts with a decrease in temperature. If the change in temperature is uniform, and if the material is isotropic and homogeneous, then all lines on the material will change dimensions by equal amounts. This will result in a normal strain, but there will be no change in the angles between any two lines, and hence there will be no shear strain produced. Experimental observations confirm this deduction. Experiments also show that the change in temperature ΔT is related to the thermal normal strain ε_T,

$$\varepsilon_T = \alpha \Delta T \tag{3.26}$$

where the Greek letter alpha α is the linear coefficient of thermal expansion. The linear relationship given by Equation (3.26) is valid for metals at temperatures well below the melting point. In this linear region the strains for most metals are small and the usual units for α are $\mu/^{\circ}$F or $\mu/^{\circ}$C, where $\mu = 10^{-6}$. Throughout the discussion in this section it is assumed that the material is in the linear region.

The tension test described in Section 3.1 is conducted at some ambient temperature. We expect the stress–strain curve to have the same character at two different ambient temperatures. If we raise the temperature by a small amount before we start the tension test then the expansion of specimen will result in a thermal strain, but there will be no stresses shifting the stress–strain curve from point O to point O_1, as shown in Figure 3.32. The total strain at any point is the sum of mechanical strain and thermal strains:

$$\boxed{\varepsilon = \frac{\sigma}{E} + \alpha \Delta T} \tag{3.27}$$

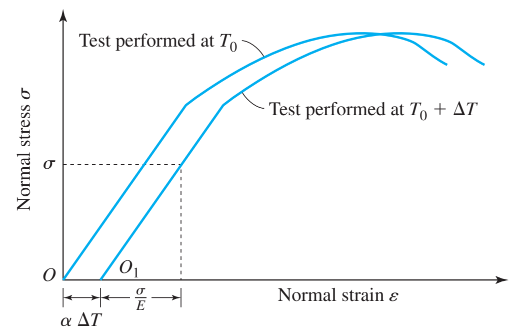


Figure 3.32 Effect of temperature on stress–strain curve.

Equation (3.27) and Figure 3.32 are valid only for small temperature changes well below the melting point. Material non-homogeneity, material anisotropy, nonuniform temperature distribution, or reaction forces from body constraints are the reasons for the generation of stresses from temperature changes. Alternatively, no thermal stresses are produced in a homogeneous, isotropic, unconstrained body due to uniform temperature changes.

The generalized Hooke's law relates mechanical strains to stresses. The total normal strain, as seen from Equation (3.27), is the sum of mechanical and thermal strains. For isotropic materials undergoing *small* changes in temperature, the generalized Hooke's law is written as shown in Equations (3.28a) through (3.28f).

Mechanical Strain | Thermal Strain

$$\varepsilon_{xx} = \frac{\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})}{E} + \alpha \Delta T \tag{3.28a}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})}{E} + \alpha \Delta T \tag{3.28b}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})}{E} + \alpha \Delta T \tag{3.28c}$$

$$\gamma_{xy} = \tau_{xy}/G \tag{3.28d}$$

$$\gamma_{yz} = \tau_{yz}/G \tag{3.28e}$$

$$\gamma_{zx} = \tau_{zx}/G \tag{3.28f}$$

Homogeneity of the material or the uniformity of the temperature change are irrelevant as Hooke's law is written for a point and not for the whole body.

EXAMPLE 3.10

A circular bar ($E = 200$ GPa, $\nu = 0.32$, and $\alpha = 11.7\ \mu/°C$) has a diameter of 100 mm. The bar is built into a rigid wall on the left, and a gap of 0.5 mm exists between the right wall and the bar prior to an increase in temperature, as shown in Figure 3.33.The temperature of the bar is increased uniformly by 80°C. Determine the average axial stress and the change in the diameter of the bar.

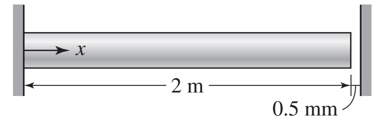


Figure 3.33 Bar in Example 3.10.

METHOD 1: PLAN

A reaction force in the axial direction will be generated to prevent an expansion greater than the gap. This would generate σ_{xx}. As there are no forces in the y or z direction, the other normal stresses σ_{yy} and σ_{zz} can be approximated to zero in Equation (3.28a). The total deformation is the gap, from which the total average axial strain for the bar can be found. The thermal strain can be calculated from the change in the given temperature. Thus in Equation (3.28a) the only unknown is σ_{xx}. Once σ_{xx} has been calculated, the strain ε_{yy} can be found from Equation (3.28b) and the change in diameter calculated.

SOLUTION

The total axial strain is the total deformation (gap) divided by the length of the bar,

$$\varepsilon_{xx} = \frac{0.5 \times 10^{-3}\ \text{m}}{2\ \text{m}} = 250 \times 10^{-6} \tag{E1}$$

$$\alpha\ \Delta T = 11.7 \times 10^{-6} \times 80 = 936 \times 10^{-6} \tag{E2}$$

Because σ_{yy} and σ_{zz} are zero, Equation (3.28a) can be written as $\varepsilon_{xx} = \sigma_{xx}/E + \alpha\ \Delta T$, from which we can obtain σ_{xx},

$$\sigma_{xx} = E(\varepsilon_{xx} - \alpha\,\Delta T) = (200 \times 10^{9}\ \text{N/m}^2)(250 - 936)10^{-6} = -137.2 \times 10^{6}\ \text{N/m}^2 \tag{E3}$$

ANS. $\sigma_{xx} = 137.2\ \text{MPa (C)}$

From Equation (3.28b) we can obtain ε_{yy} and calculate the change in diameter,

$$\varepsilon_{yy} = -\nu\frac{\sigma_{xx}}{E} + \alpha\ \Delta T = -0.25\left(\frac{-137.2 \times 10^{6}\ \text{N/m}^2}{200 \times 10^{9}\ \text{N/m}^2}\right) + 936 \times 10^{-6} = 1.107 \times 10^{-3} \tag{E4}$$

$$\Delta D = \varepsilon_{yy}D = 1.107 \times 10^{-3} \times 100\ \text{mm} \tag{E5}$$

ANS. $\Delta D = 0.1107$ mm increase

COMMENTS

1. If $\alpha\,\Delta T$ were less than ε_{xx}, then σ_{xx} would come out as tension and our assumption that the gap closes would be invalid. In such a case there would be no stress σ_{xx} generated.
2. The increase in diameter is due partly to Poisson's effect and partly to thermal strain in the y direction.

METHOD 2: PLAN

We can think of the problem in two steps: (i) Find the thermal expansion δ_T initially ignoring the restraining effect of the right wall. (ii) Apply the force P to bring the bar back to the restraint position due to the right wall and compute the corresponding stress.

SOLUTION

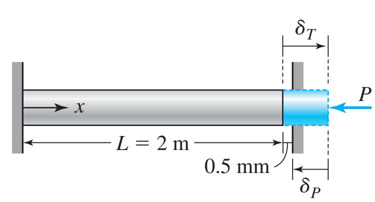


Figure 3.34 Approximate deformed shape of the bar in Example 3.10.

We draw an approximate deformed shape of the bar, assuming there is no right wall to restrain the deformation as shown in Figure 3.34.The thermal expansion δ_T is the thermal strain multiplied by the length of the bar,

$$\delta_T = (\alpha\,\Delta T)L = 11.7 \times 10^{-6} \times 80 \times 2 \text{ m} = 1.872 \times 10^{-3} \text{ m} \tag{E6}$$

We obtain the contraction δ_P to satisfy the restraint imposed by the right wall by subtracting the gap from the thermal expansion.

$$\delta_P = \delta_T - 0.5 \times 10^{-3} \text{ m} = 1.372 \times 10^{-3} \text{ m} \tag{E7}$$

We can then find the mechanical strain and compute the corresponding stress:

$$\varepsilon_P = \frac{\delta_P}{L} = \frac{1.372 \times 10^{-3} \text{ m}}{2 \text{ m}} = 0.686 \times 10^{-3} \tag{E8}$$

$$\sigma_P = E\varepsilon_P = (200 \times 10^9 \text{ N/m}^2) \times 0.686 \times 10^{-3} = 137.2 \times 10^6 \text{ N/m}^2 \tag{E9}$$

ANS. $\sigma_P = 137.2 \text{ MPa } (C)$

The change in diameter can be found as in Method 1.

COMMENT

In Method 1 we ignored the intermediate steps and conducted the analysis at equilibrium. We implicitly recognized that for a linear system the process of reaching equilibrium is immaterial. In Method 2 we conducted the thermal and mechanical strain calculations separately. Method 1 is more procedural. Method 2 is more intuitive.

EXAMPLE 3.11

Solve Example 3.8 with a temperature increase of 20°C. Use $\alpha = 23\ \mu/°\text{C}$.

PLAN

Shear stress is unaffected by temperature change and its value is the same as in Example 3.8. Hence $\tau_{xy} = 21$ MPa. In Equations 3.28a and 3.28b $\sigma_{zz} = 0$, $\varepsilon_{xx} = 650\ \mu$, and $\varepsilon_{yy} = 300\ \mu$ are known and $\alpha\Delta T$ can be found and substituted to generate two equations in the two unknown stresses σ_{xx} and σ_{yy}, which are found by solving the equations simultaneously. Then from Equation (3.28c), the normal strain ε_{zz} can be found.

SOLUTION

We can find the thermal strain as $\Delta T = 20$ and $\alpha\Delta T = 460 \times 10^{-6}$. Equations 3.28a and 3.28b and can be rewritten with $\sigma_{zz} = 0$,

$$\sigma_{xx} - \nu\sigma_{yy} = E(\varepsilon_{xx} - \alpha\Delta T) = (70 \times 10^9 \text{ N/m}^2)(650 - 460)10^{-6} \quad \text{or}$$

$$\sigma_{xx} - 0.25\sigma_{yy} = 13.3 \text{ MPa} \tag{E1}$$

$$\sigma_{yy} - \nu\sigma_{xx} = E(\varepsilon_{yy} - \alpha\Delta T) = (70 \times 10^9 \text{ N/m}^2)(300 - 460)10^{-6} \quad \text{or}$$

$$\sigma_{yy} - 0.25\sigma_{xx} = -11.2 \text{ MPa} \tag{E2}$$

By solving Equations (E1) and (E2) we obtain σ_{xx} and σ_{yy}.

ANS. $\sigma_{xx} = 11.2 \text{ MPa(T)} \qquad \sigma_{yy} = 8.4 \text{ MPa(C)}$

From Equation (3.28c) with $\sigma_{zz} = 0$ we obtain

$$\varepsilon_{zz} = \frac{-\nu(\sigma_{xx} + \sigma_{yy})}{E} + \alpha\,\Delta T = \frac{-0.25(11.2 - 8.4)(10^6) \text{ N/m}^2}{(70 \times 10^9 \text{ N/m}^2)} + 460 \times 10^{-6} \tag{E3}$$

ANS. $\varepsilon_{zz} = 460\ \mu$

COMMENT

1. Equations (E1) and (E2) once more have the same structure as in Example 3.8. The only difference is that in Example 3.8 we were given the mechanical strain and in this example we obtained the mechanical strain by subtracting the thermal strain from the total strain.

PROBLEM SET 3.3

Stress concentration

3.119 A steel bar is axially loaded, as shown in Figure P3.119. Determine the factor of safety for the bar if yielding is to be avoided. The normal yield stress for steel is 30 ksi. Use the stress concentration factor chart in Section C.4 in Appendix.

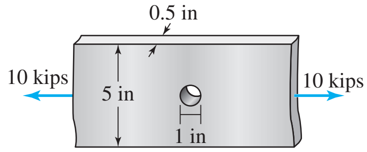


Figure P3.119

3.120 The stress concentration factor for a stepped flat tension bar with shoulder fillets shown in Figure P3.120 was determined as given by the equation below. The equation is valid only if $H/d > 1 + 2r/d$ and $L/H > 5.784 - 1.89r/d$. The nominal stress is P/dt. Make a chart for the stress concentration factor versus H/d for the following values of r/d: 0.2, 0.4, 0.6, 0.8, 1.0. (Use of a spreadsheet is recommended.)

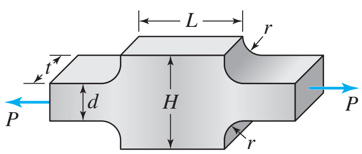


Figure P3.120

$$K_{conc} = 1.970 - 0.384\left(\frac{2r}{H}\right) - 1.018\left(\frac{2r}{H}\right)^2 + 0.430\left(\frac{2r}{H}\right)^3$$

3.121 Determine the maximum normal stress in the stepped flat tension bar shown in Figure P3.120 for the following data: $P = 9$ kips, $H = 8$ in, $d = 3$ in, $t = 0.125$ in, and $r = 0.625$ in.

3.122 An aluminum stepped tension bar is to carry a load $P = 56$ kN. The normal yield stress of aluminum is 160 MPa. The bar in Figure P3.120 has $H = 300$ mm, $d = 100$ mm, and $t = 10$ mm. For a factor of safety of 1.6, determine the minimum value r of the fillet radius if yielding is to be avoided.

3.123 The stress concentration factor for a flat tension bar with U-shaped notches shown in Figure P3.123 was determined as given by the equation below. The nominal stress is P/Ht. Make a chart for the stress concentration factor vs. r/d for the following values of H/d: 1.25, 1.50, 1.75, 2.0. (Use of a spreadsheet is recommended.)

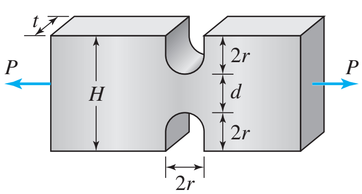


Figure P3.123

$$K_{conc} = 3.857 - 5.066\left(\frac{4r}{H}\right) + 2.469\left(\frac{4r}{H}\right)^2 - 0.258\left(\frac{4r}{H}\right)^3$$

3.124 Determine the maximum normal stress in the flat tension bar shown in Figure P3.123 for the following data: $P = 150$ kN, $H = 300$ mm, $r = 15$ mm, and $t = 5$ mm.

3.125 A steel tension bar with U-shaped notches of the type shown in Figure P3.123 is to carry a load $P = 18$ kips. The normal yield stress of steel is 30 ksi. The bar has $H = 9$ in., d = 6 in. and $t = 0.25$ in. For a factor of safety of 1.4, determine the value of r if yielding is to be avoided.

Temperature effects

3.126 An iron rim ($\alpha = 6.5\ \mu/°F$) of 35.98-in diameter is to be placed on a wooden cask of 36-in. diameter. Determine the minimum temperature increase needed to slip the rim onto the cask.

3.127 The temperature is increased by 60°C in both steel (E_s = 200 GPa, α_s = 12.0 μ/°C) and aluminum (E = 72 GPa, α = 23.0 μ/°C). Determine the angle by which the pointer rotates from the vertical position (Figure P3.127).

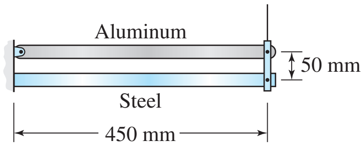


Figure P3.127

3.128 Solve Problem 3.73 if the temperature decrease is 25°C. Use α = 11.7 μ/°C.

3.129 Solve Problem 3.74 if the temperature increase is 50°C. Use α = 23.6 μ/°C.

3.130 Solve Problem 3.81 if the temperature increase is 40°F. Use α = 6.5 μ/°F.

3.131 Solve Problem 3.82 if the temperature decrease is 100°F. Use α = 12.8 μ/°F.

3.132 Solve Problem 3.83 if the temperature decrease is 75°C. Use α = 26.0 μ/°C.

3.133 A plate (E = 30,000 ksi, ν = 0.25, $\alpha = 6.5 \times 10^{-6}$/°F) cannot expand in the y direction and can expand at most by 0.005 in. in the x direction, as shown in Figure P3.133. Assuming plane stress, determine the average normal stresses in the x and y directions due to a uniform temperature increase of 100°F.

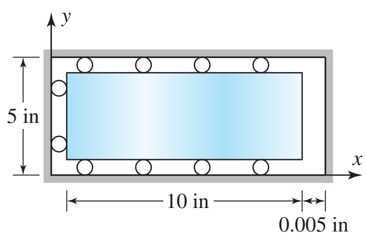


Figure P3.133

3.134 Derive the following relations of normal stresses in terms of normal strains from Equations (3.28a), (3.28b), and (3.28c):

$$\begin{aligned}
\sigma_{xx} &= [(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}]\frac{E}{(1-2\nu)(1+\nu)} - \frac{E\alpha\,\Delta T}{1-2\nu} \\
\sigma_{yy} &= [(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz} + \nu\varepsilon_{xx}]\frac{E}{(1-2\nu)(1+\nu)} - \frac{E\alpha\,\Delta T}{1-2\nu} \\
\sigma_{zz} &= [(1-\nu)\varepsilon_{zz} + \nu\varepsilon_{xx} + \nu\varepsilon_{yy}]\frac{E}{(1-2\nu)(1+\nu)} - \frac{E\alpha\,\Delta T}{1-2\nu}
\end{aligned} \qquad (3.29)$$

3.135 For a point in plane stress show that

$$\sigma_{xx} = (\varepsilon_{xx} + \nu\varepsilon_{yy})\frac{E}{1-\nu^2} - \frac{E\alpha\,\Delta T}{1-\nu} \qquad \sigma_{yy} = (\varepsilon_{yy} + \nu\varepsilon_{xx})\frac{E}{1-\nu^2} - \frac{E\alpha\,\Delta T}{1-\nu} \qquad (3.30)$$

3.136 For a point in plane stress show that

$$\varepsilon_{zz} = -\left(\frac{\nu}{1-\nu}\right)(\varepsilon_{xx} + \varepsilon_{yy}) + \left(\frac{1+\nu}{1-\nu}\right)\alpha\,\Delta T \qquad (3.31)$$

3.10* FATIGUE

Try to break a piece of wire (such as a paper clip) by pulling on it by hand. You will not be able to break because you would need to exceed the ultimate stress of the material. Next take the same piece of wire and bend it one way and then the other a few times, and you will find that it breaks easily. The difference is the phenomena of fatigue.

All materials are *assumed* to have microcracks. These small crack length are not critical and are averaged as ultimate strength for the bulk material in a tension test. However, if the material is subjected to cyclic loading, these microcracks can grow until a crack reaches some critical length, at which time the remaining material breaks. The stress value at rupture in a cyclic loading is significantly lower than the ultimate stress of the material. Failure due to cyclic loading at stress levels significantly lower than the static ultimate stress is called **fatigue**.

Failure due to fatigue is like a brittle failure, irrespective of whether the material is brittle or ductile. There are two phases of failure. In the first phase the microcracks grow. These regions of crack growth can be identified by striation marks, also called beach marks, as shown in Figure 3.35. On examination of a fractured surface, this region of microcrack growth shows only small deformation. In phase 2, which is after the critical crack length has been reached, the failure surface of the region shows significant deformation.

Striation Marks

Figure 3.35 Failure of lead solder due to fatigue. (Courtesy Professor I. Miskioglu.)

The following strategy is used in design to account for fatigue failure. Experiments are conducted at different magnitude levels of cyclic stress, and the number of cycles at which the material fails is recorded. There is always significant scatter in the data. At low level of stress the failure may occur in millions and, at times, billions of cycles. To accommodate this large scale, a log scale is used for the number of cycles. A plot is made of stress versus the number of cycles to failure called the ***S–N* curve** as shown in Figure 3.36. Notice that the curve approaches a stress level asymtotically, implying that if stresses are kept below this level, then the material would not fail under cyclic loading. The highest stress level for which the material would not fail under cyclic loading is called **endurance limit** or **fatigue strength**.

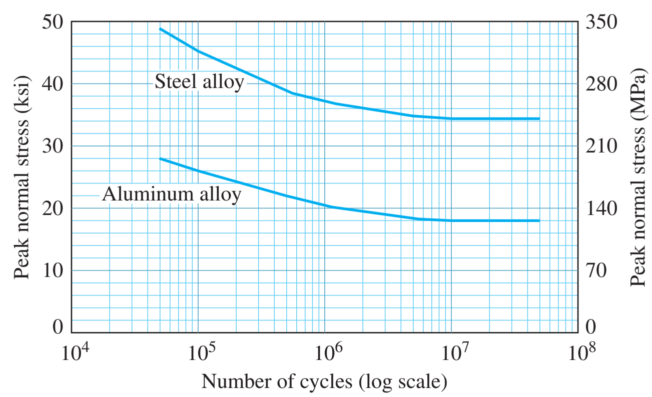


Figure 3.36 *S–N* curves.

It should be emphasized that a particular *S–N* curve for a material depends on many factors, such as manufacturing process, machining process, surface preparation, and operating environment. Thus two specimens made from the same steel alloys, but with a different history, will result in different *S–N* curves. Care must be taken to use an *S–N* curve that corresponds as closely as possible to the actual situation.

In a typical preliminary design, static stress analysis would be conducted using the peak load of the cyclic loading. Using an appropriate *S–N* curve, the number of cycles to failure for the peak stress value is calculated. This number of cycles to failure is the predicted life of the structural component. If the predicted life is unacceptable, then the component will be redesigned to lower the peak stress level and hence increase the number of cycles to failure.

EXAMPLE 3.12

The steel plate shown in Figure 3.37 has the *S–N* curve given in Figure 3.36. (a) Determine the maximum diameter of the hole to the nearest millimeter if the predicted life of one-half million cycles is desired for a uniform far-field stress $\sigma = 75$ MPa. (b) For the hole radius in part (a), what percentage reduction in far-field stress must occur if the predicted life is to increase to 1 million cycles?

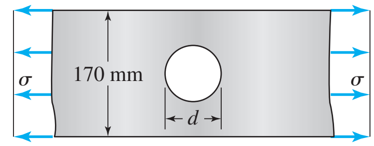


Figure 3.37 Uniaxially loaded plate with a hole in Example 3.12.

PLAN

(a) From Figure 3.36 we can find the maximum stress that the material can carry for one-half million cycles. From Equation (3.25) the gross stress concentration factor K_{gross} can be found. From the plot of K_{gross} in Figure A.13 of Appendix C we can estimate the ratio d/H and find the diameter d of the hole. (b) The percentage reduction in the gross nominal stress σ is the same as that in the maximum stress values in Figure 3.36, from one million cycles to one-half million cycles.

SOLUTION

(a) From Figure 3.36 the maximum allowable stress for one-half million cycles is estimated as 273 MPa. From Equation (3.25) the gross stress concentration factor is

$$K_{gross} = \frac{\sigma_{max}}{\sigma_{nominal}} = \frac{273 \text{ MPa}}{75 \text{ MPa}} = 3.64 \tag{E1}$$

From Figure A.13 of Appendix C the value of the ratio d/H corresponding to $K_{gross} = 3.64$ is 0.374.

$$d = 0.374 \times H = 0.374 \times 170 \text{ mm} = 63.58 \text{ mm} \tag{E2}$$

The maximum permissible diameter to the nearest millimeter can be obtained by rounding downward.

ANS. $d_{max} = 63$ mm

(b) From Figure 3.36 the maximum allowable stress for one million cycles is estimated as 259 MPa. Thus the percentage reduction in maximum allowable stress is [(273MPa – 259 MPa)/273 MPa](100) = 5.13%. As the geometry is the same as in part (a), the percentage reduction in far-field stress should be the same as in the maximum allowable stress.

ANS. The percentage reduction required is 5.13%.

COMMENT

1. A 5.13% reduction in peak stress value causes the predicted life cycle to double. Many factors can cause small changes in stress values, resulting in a very wide range of predictive life cycles. Examples include our estimates of the allowable stress in Figure 3.36, of the ratio d/H from Figure A.13 of Appendix C, of the far-field stress σ, and the tolerances of drilling the hole. Each is factor that can significantly affect our life prediction of the component. This emphasizes that the data used in predicting life cycles and failure due to fatigue must be of much higher accuracy than in traditional engineering analysis.

MoM in Action: The Comet / High Speed Train Accident

On January 10, 1954, the *de Havilland Comet* failed in midair near the Italian island Elba, killing all 35 people on board. On June 3, 1998, near the village of Eschede in Germany, a high-speed train traveling at nearly 200 km/h derailed, killing 101 people and injuring another 88. The events are a cautionary tale about the inherent dangers of new technologies and the high price of knowledge.

The *Comet* represented state-of-the-art technology. Passengers had a pressurized air cabin and slightly rounded square windows (Figure 1.39a) to look outside. The world's first commercial jet airliner flew 50% faster than the piston-engine aircrafts of that time, reducing flight times. It also flew higher, above adverse weather, for greater fuel efficiency and fewer vibrations. Its advanced aluminum alloy was postcard thin, to reduce weight, and adhesively bonded, lowering the risk of cracks spreading from rivets. Stress cycling due to pressurizing and depressurizing on plane that flies to 36,000 feet and returns to ground was simulated on a design prototype using a water tank. The plane was deemed safe for at least 16,000 flights.

On January 22, 1952, the *Comet* received a certificate of airworthiness. It crashed less than two years later after only 1290 flights, and the initial investigation failed to determine why. Flights resumed March 23, 1954, but on April 8, a second *Comet* crashed near Naples on its way from Rome to Cairo – after only 900 flights. Once more flights were grounded, while pressurizing and depressurizing testing was conducted on a plane that had gone through 1221 flights. It failed the tests after 1836 additional simulations.

Why did the initial testing on the prototype give such misleading results? Stresses near the window corners were far in excess of expectations, resulting in shorter fatigue life. Unlike static, fatigue test results should be used with great caution in extrapolating to field conditions. Passenger windows were made elliptical in shape, for a lower stress concentration. With this and other design improvements, *Comets* were used by many airlines for the next 30 years.

(a)

(b)

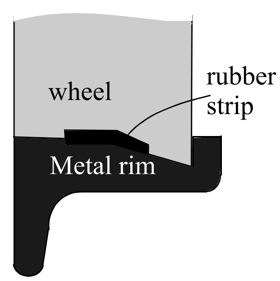


Figure 3.38 (a) de Havilland Comet 1 (b) Cross-section of high speed train wheel.

The high-speed intercity express (ICE) was the pride of the German railways. The first generation of these trains had single-cast wheels. At cruising speed, wheels deformation was causing vibrations. The wheels were redesigned with a rubber damping strip with a metal rim, as shown in Figure 1.39b. This design, already in use in streetcars, resolved the vibrations. However, the metal rims were failing earlier then predicted by design. The railway authority had noticed the problem long before the accident, but decided to merely replace the wheels more often. The decision proved disastrous.

Six kilometers from Eschede, the wheel rim from one axle peeled and punctured the floor. The train derailed in minutes. And investigation established that the rims become thinner owing to wear, and fatigue-induced cracks can cause failure earlier than the design prediction. The wheel design is now once more single cast, and alternative solutions to the vibration problems were found. Today the high-speed ICE is used for much of Germany.

No laboratory test can accurately predict fatigue life cycles under field conditions. Regular inspection of planes and high-speed train wheels for fatigue cracks is now standard practice.

3.11* NONLINEAR MATERIAL MODELS

Rubber, plastics, muscles, and other organic tissues exhibit nonlinearity in the stress–strain relationship, even at small strains. Metals also exhibit nonlinearity after yield stress. In this section we consider various nonlinear material models— the equations that represent the stress–strain nonlinear relationship. The material constants in the equations are found by least-squares fit of the stress–strain equation to the experimental data. For the sake of simplicity we shall assume that the material behavior is the same in tension and in compression.

We will consider three material models that are used in analytical and numerical analysis:

1. The *elastic–perfectly plastic* model, in which the nonlinearity is approximated by a constant.
2. The *linear strain-hardening* (or *bilinear*) model, in which the nonlinearity is approximated by a linear function.
3. The *power law* model, in which the nonlinearity is approximated by a one-term nonlinear function.

Other material models are described in the problems. The choice of material model depends not only on the material stress–strain curve, but also on the need for accuracy and the resulting complexity of analysis.

3.11.1 Elastic–Perfectly Plastic Material Model

Figure 3.39 shows the stress–strain curves describing an elastic–perfectly plastic behavior of a material. It is assumed that the material has the same behavior in tension and in compression. Similarly, for shear stress–strain, the material behavior is the same for positive and negative stresses and strains.

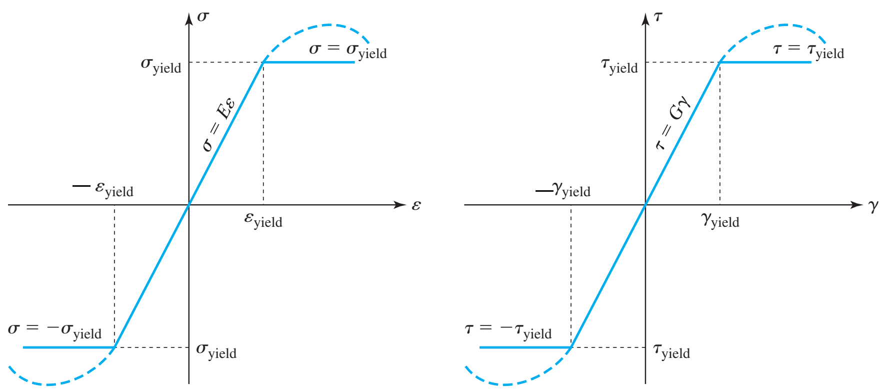


Figure 3.39 Elastic–perfectly plastic material behavior.

Before yield stress the stress–strain relationship is given by Hooke's law. After yield stress the stress is a constant. The elastic–perfectly plastic material behavior is a simplifying approximation[2] used to conduct an elastic–plastic analysis. The approximation is conservative in that it ignores the material capacity to carry higher stresses than the yield stress. The stress–strain curve are given by

$$\sigma = \begin{cases} \sigma_{yield}, & \varepsilon \geq \varepsilon_{yield} \\ E\varepsilon, & -\varepsilon_{yield} \leq \varepsilon \leq \varepsilon_{yield} \\ -\sigma_{yield}, & \varepsilon \leq -\varepsilon_{yield} \end{cases} \tag{3.32}$$

$$\tau = \begin{cases} \tau_{yield}, & \gamma \geq \gamma_{yield} \\ G\gamma, & -\gamma_{yield} \leq \gamma \leq \gamma_{yield} \\ -\tau_{yield}, & \gamma \leq -\gamma_{yield} \end{cases} \tag{3.33}$$

The set of points forming the boundary between the elastic and plastic regions on a body is called **elastic–plastic boundary.** Determining the location of the elastic–plastic boundary is one of the critical issues in elastic–plastic analysis. The examples will show, the location of the elastic–plastic boundary is determined using two observations:

[2]Limit analysis is a technique based on elastic-plastic material behavior. It can be used to predict the maximum load a complex structure like a truss can support.

1. On the elastic–plastic boundary, the strain must be equal to the yield strain, and stress equal to yield stress. Deformations and strains are continuous at all points, including points at the elastic–plastic boundary.
2. If deformation is not continuous, then it is implied that holes or cracks are being formed in the material. If strains, which are derivative displacements, are not continuous, then corners are being formed during deformation.

3.11.2 Linear Strain-Hardening Material Model

Figure 3.40 shows the stress–strain curve for a linear strain-hardening model, also referred to as *bilinear* material[3] model. It is assumed that the material has the same behavior in tension and in compression. Similarly, for shear stress and strain, the material behavior is the same for positive and negative stresses and strains.

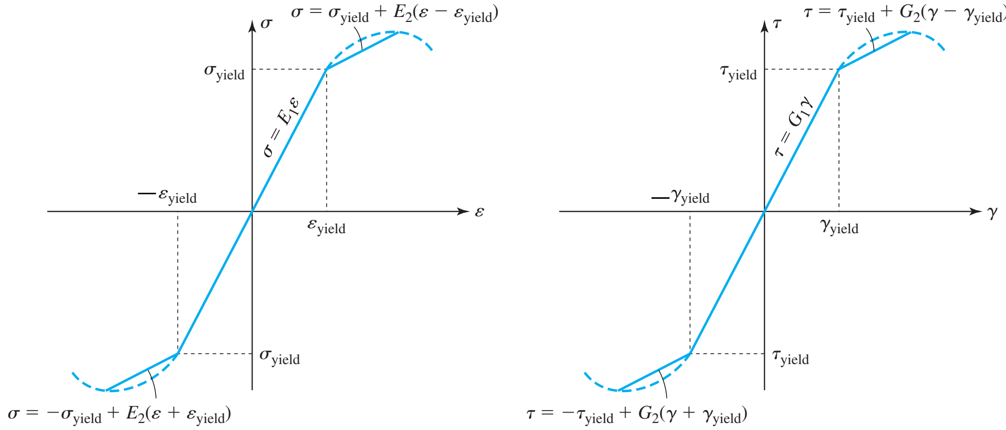


Figure 3.40 Linear strain-hardening model.

This is another conservative, simplifying approximation of material behavior: we once more ignore the material ability to carry higher stresses than shown by straight lines. The location of the elastic–plastic boundary is once more a critical issue in the analysis, and it is determined as in the previous section.

The stress–strain curves are given by

$$\sigma = \begin{cases} \sigma_{yield} + E_2(\varepsilon - \varepsilon_{yield}) & \varepsilon \ge \varepsilon_{yield} \\ E_1\varepsilon & -\varepsilon_{yield} \le \varepsilon \le \varepsilon_{yield} \\ -\sigma_{yield} + E_2(\varepsilon + \varepsilon_{yield}) & \varepsilon \ge \varepsilon_{yield} \end{cases} \tag{3.34}$$

$$\tau = \begin{cases} \tau_{yield} + G_2(\gamma - \gamma_{yield}), & \gamma \ge \gamma_{yield} \\ G_1\gamma, & -\gamma_{yield} \le \gamma \le \gamma_{yield} \\ -\tau_{yield} + G_2(\gamma + \gamma_{yield}), & \gamma \le \gamma_{yield} \end{cases} \tag{3.35}$$

3.11.3 Power-Law Model

Figure 3.41 shows a power-law representation of a nonlinear stress–strain curve. It is assumed that the material has the same behavior in tension and in compression. Similarly for shear stress and strain; the material behavior is the same for positive and negative stresses and strains. The stress–strain curve are given by

$$\sigma = \begin{cases} E\varepsilon^n, & \varepsilon \ge 0 \\ -E(-\varepsilon)^n, & \varepsilon < 0 \end{cases} \qquad \tau = \begin{cases} G\gamma^n, & \gamma \ge 0 \\ -G(-\gamma)^n, & \gamma < 0 \end{cases} \tag{3.36}$$

[3]Incremental plasticity is a numerical technique that approximated the non-linear stress-strain curve by series of straight lines over small intervals.

The constants E and G are the strength coefficients, and n is the strain-hardening coefficient. They are determined by least-squares fit to the experimental stress–strain curve. Materials such as most metals in plastic region or most plastics are represented by the solid curve with the strain-hardening coefficient less than one. Materials like soft rubber, muscles and other organic materials are represented by the dashed line with a strain-hardening coefficient greater than one.

From Equation (3.36) we note that when strain is negative, the term in parentheses becomes positive, permitting evaluation of the number to fractional powers. Furthermore with negative strain we obtain negative stress, as we should.

In Section 3.11.2 we saw that the stress–strain relationship could be written using different equations for different stress levels. We could, in a similar manner, combine a linear equation for the linear part and a nonlinear equation for the nonlinear part, or we could combine two nonlinear equations, thus creating additional material models. Other material models are considered in the problems.

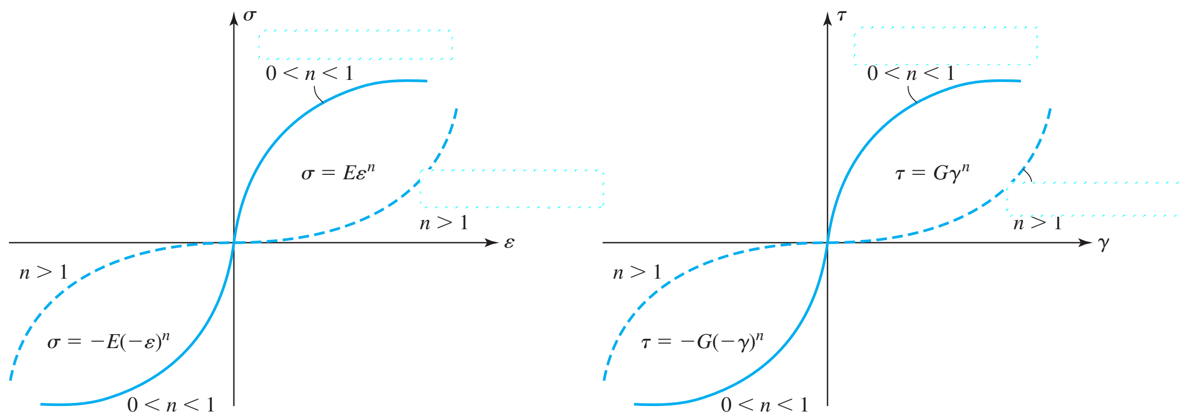


Figure 3.41 Nonlinear stress–strain curves.

EXAMPLE 3.13

Aluminum has a yield stress $\sigma_{\text{yield}} = 40$ ksi in tension, a yield strain $\varepsilon_{\text{yield}} = 0.004$, an ultimate stress $\sigma_{\text{ult}} = 45$ ksi, and the corresponding ultimate strain $\varepsilon_{\text{ult}} = 0.17$. Determine the material constants and plot the corresponding stress–strain curves for the following models: (a) the elastic–perfectly plastic model. (b) the linear strain-hardening model. (c) the nonlinear power-law model.

PLAN

We have coordinates of three points on the curve: P_0 ($\sigma_0 = 0.00$, $\varepsilon_0 = 0.000$), P_1 ($\sigma_1 = 40.0$, $\varepsilon_1 = 0.004$), and P_2 ($\sigma_2 = 45.0$, $\varepsilon_2 = 0.017$). Using these data we can find the various constants in the material models.

SOLUTION

(a) The modulus of elasticity E is the slope between points P_0 and P_1. After yield stress, the stress is a constant. The stress–strain behavior can be written as

$$E_1 = \frac{\sigma_1 - \sigma_0}{\varepsilon_1 - \varepsilon_0} = \frac{40 \text{ ksi}}{0.004} = 10{,}000 \text{ ksi} \tag{E1}$$

$$\textbf{ANS.} \qquad \sigma = \begin{cases} 10{,}000\varepsilon \text{ ksi}, & |\varepsilon| \le 0.004 \\ 40 \text{ ksi}, & |\varepsilon| \ge 0.004 \end{cases} \tag{E2}$$

(b) In the linear strain-hardening model the slope of the straight line before yield stress is as calculated in part (a). After the yield stress, the slope of the line can be found from the coordinates of points P_1 and P_2. The stress–strain behavior can be written as

$$E_2 = \frac{\sigma_2 - \sigma_1}{\varepsilon_2 - \varepsilon_1} = \frac{5 \text{ ksi}}{0.013} = 384.6 \text{ ksi} \tag{E3}$$

$$\textbf{ANS.} \qquad \sigma = \begin{cases} 10{,}000\varepsilon \text{ ksi}, & |\varepsilon| \le 0.004 \\ 40 + 384.6(\varepsilon - 0.004) \text{ ksi}, & |\varepsilon| \ge 0.004 \end{cases} \tag{E4}$$

(c) The two constants E and n in $\sigma = E\varepsilon^n$ can be found by substituting the coordinates of the two point P_1 and P_2, to generate

$$40 = E(0.004)^n \quad \text{(E5)}$$

$$45 = E(0.017)^n \quad \text{(E6)}$$

Dividing Equation (E6) by Equation (E5) and taking the logarithm of both sides, we solve for n:

$$\ln\left(\frac{0.017}{0.004}\right)^n = \ln\left(\frac{45}{40}\right) \quad or \quad n\ \ln(4.25) = \ln(1.125) \quad \text{or} \quad n = 0.0814 \quad \text{(E7)}$$

Substituting Equation (E7) into Equation (E5), we obtain the value of E:

$$E = (40\ \text{ksi})/(0.004)^{0.0814} = 62.7\ \text{ksi} \quad \text{(E8)}$$

We can now write the stress-strain equations for the power law model.

$$\textbf{ANS.} \quad \sigma = \begin{cases} 62.7\varepsilon^{0.0814}\ \text{ksi} & \varepsilon \geq 0 \\ -62.7(-\varepsilon)^{0.0814}\ \text{ksi} & \varepsilon < 0 \end{cases} \quad \text{(E9)}$$

Stresses at different strains can be found using Equations (E2), (E4), and (E9) and plotted as shown in Figure 3.42.

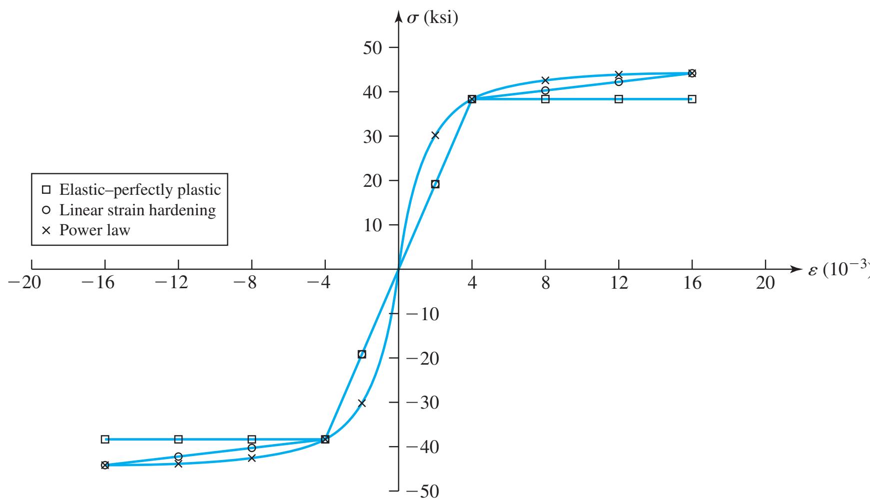


Figure 3.42 Stress–strain curves for different models in Example 3.13.

EXAMPLE 3.14

On a cross-section of a hollow circular shaft shown in Figure 3.43, the shear strain in polar coordinates was found to be $\gamma_{x\theta} = 3\rho \times 10^{-3}$, where ρ is the radial coordinate measured in inches. Write expressions for $\tau_{x\theta}$ as a function of ρ, and plot the shear strain γ_{xq} and shear stress τ_{xq} distributions across the cross section. Assume the shaft is made from elastic–perfectly plastic material that has a yield stress $\tau_{\text{yield}} = 24$ ksi and a shear modulus $G = 6000$ ksi.

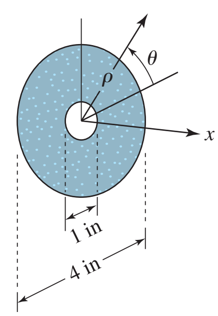


Figure 3.43 Hollow shaft in Example 3.14.

PLAN

The yield strain in shear γ_{yield} can be found from the yield stress τ_{yield} and the shear modulus G. The location of the elastic–plastic boundary ρ_y can be determined at which the given shear strain reaches the value of γ_{yield}. The shear stress at points before ρ_y can be found from Hooke's law, and after ρ_y it will be the yield stress.

SOLUTION

The location of the elastic–plastic boundary can be found as

$$\gamma_{yield} = \frac{\tau_{yield}}{G} = \frac{24 \times 10^3 \text{ ksi}}{6000 \times 10^3 \text{ ksi}} = 0.004 = 0.003\rho_y \qquad or \qquad \rho_y = \frac{0.004}{0.003} = 1.33 \text{ in.} \tag{E1}$$

Up to ρ_y stress and strain are related by Hooke's law, and hence

$$\tau_{x\theta} = G\gamma_{x\theta} = (6 \times 10^6 \text{ psi}) \times 3\rho \times 10^{-3} = 18\rho \times 10^3 \text{ psi} \tag{E2}$$

After ρ_y the stress is equal to τ_y, and the shear stress can be written as

ANS. $$\tau_{x\theta} = \begin{cases} 18\rho \text{ ksi}, & 0.5 \text{ in.} \le \rho \le 1.333 \text{ in.} \\ 24 \text{ ksi}, & 1.333 \text{ in.} \le \rho \le 2.0 \text{ in.} \end{cases} \tag{E3}$$

The shear strain and shear stress distributions across the cross section are shown in Figure 3.44.

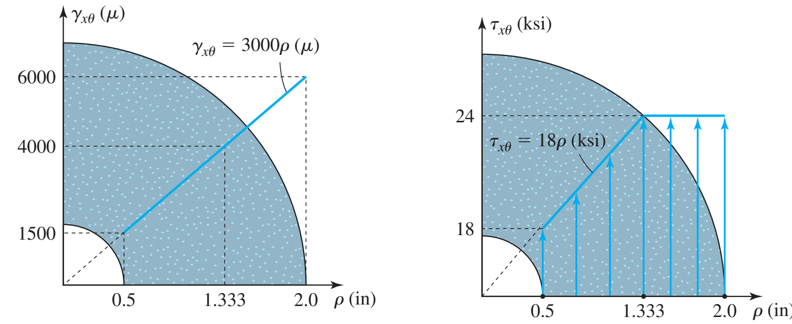


Figure 3.44 Strain and stress distributions.

COMMENT

1. In this problem we knew the strain distribution and hence could locate the elastic–plastic boundary easily. In most problems we do not know the strains due to a load, and finding the elastic–plastic boundary is significantly more difficult.

EXAMPLE 3.15

Resolve Example 3.14 assuming the shaft material has a stress–strain relationship given by $\tau = 450\gamma^{0.75}$ ksi.

PLAN

Substituting the strain expression into the stress–strain equation we can obtain stress as a function of ρ and plot it.

SOLUTION

Substituting the strain distribution into the stress–strain relation we obtain

$$\tau_{x\theta} = 450 \times 0.003^{0.75}\rho^{0.75} \tag{E1}$$

ANS. $\tau_{x\theta} = 5.768\rho^{0.75}$ ksi

The shear stress can be found at several points and plotted as shown in Figure 3.45.

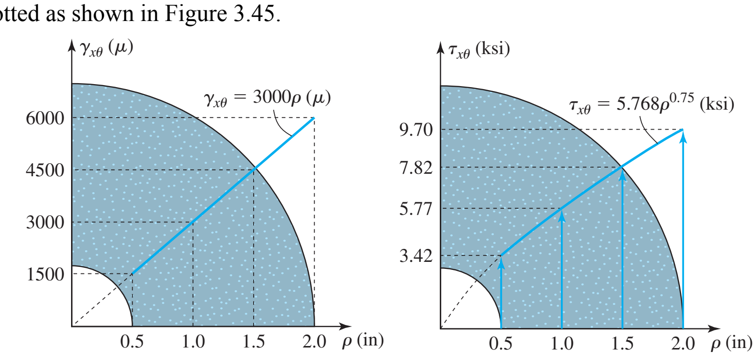


Figure 3.45 Strain and stress distributions in Example 3.15

COMMENT

1. We see that although the strain distribution is linear across the cross section, the stress distribution is nonlinear due to material nonlinearity. Deducing the stress distribution across the cross section would be difficult, but deducing a linear strain distribution is possible from geometric considerations, as will be seen in Chapter 5 for the torsion of circular shafts.

EXAMPLE 3.16

At a cross section of a beam shown in Figure 3.46, the normal strain due to bending about the z axis was found to vary as $\varepsilon_{xx} = -0.0125y$, with y measured in meters. Write the expressions for normal stress σ_{xx} as a function of y and plot the σ_{xx} distribution across the cross section. Assume the beam is made from elastic–perfectly plastic material that has a yield stress $\sigma_{yield} = 250$ MPa and a modulus of elasticity $E = 200$ GPa.

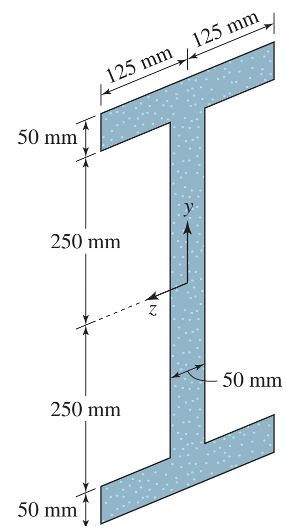


Figure 3.46 Beam cross section in Example 3.16.

PLAN

Points furthest from the origin will be the most strained, and the plastic zone will start from the top and bottom and move inward symmetrically. We can determine the yield strain ε_{yield} from the given yield stress σ_{yield} and the modulus of elasticity E. We can then find the location of the elastic–plastic boundary by finding y_y at which the normal strain reaches the value of ε_{yield}. The normal stress before y_y can be found from Hooke's law, and after y_y it will be the yield stress.

SOLUTION

The location of the elastic–plastic boundary is given by

$$\varepsilon_{yield} = \frac{\sigma_{yield}}{E} = \frac{(\pm 250 \times 10^6)\ \text{N/m}^2}{200 \times 10^9\ \text{N/m}^2} = \pm 1.25 \times 10^{-3} = -0.0125 y_y \qquad \text{or}$$

$$y_y = \frac{\pm 1.25 \times 10^{-3}}{-0.0125} = \mp 0.1\ \text{m} \tag{E1}$$

Up to the elastic–plastic boundary, i.e., y_y, the material is in the linear range and Hooke's law applies. Thus,

$$\sigma_{xx} = (200 \times 10^9\ \text{N/m}^2)(-0.0125y) = -2500y\ \text{MPa} \qquad |y| \le 0.1\ \text{m} \tag{E2}$$

The normal stress as a function of y can be written as

ANS.
$$\sigma_{xx} = \begin{cases} -250\ \text{MPa}, & 0.1\ \text{m} \le y \le 0.3\ \text{m} \\ -2500y\ \text{MPa}, & -0.1\ \text{m} \le y \le 0.1\ \text{m} \\ 250\ \text{MPa}, & -0.3\ \text{m} \le y \le -0.1\ \text{m} \end{cases} \tag{E3}$$

The normal strain and stress as a function of y can be plotted as shown in Figure 3.47

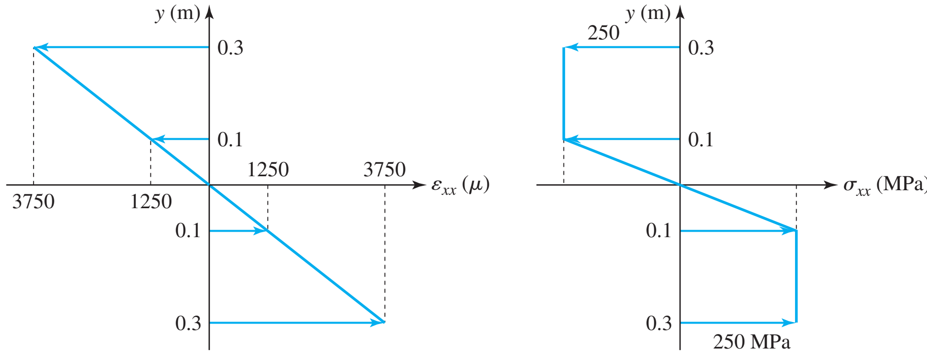


Figure 3.47 Strain and stress distributions in Example 3.16

COMMENTS

1. To better appreciate the stress distribution we can plot it across the entire cross section, as shown in Figure 3.48.

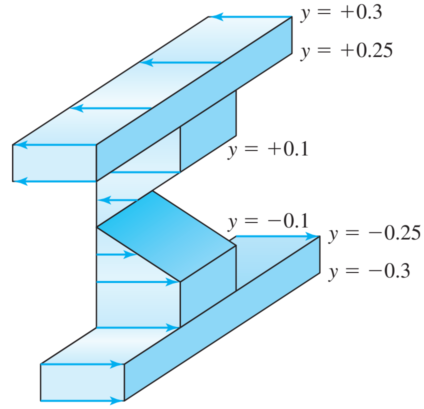


Figure 3.48 Stress distribution across cross section in Example 3.16

2. Once more we see that the stress distribution across the cross section will be difficult to deduce, but as will be seen in Chapter 6 for the symmetric bending of beams, we can deduce the approximate strain distribution from geometric considerations.

EXAMPLE 3.17

Resolve Example 3.16 assuming that the stress–strain relationship is given by $\sigma = 9000\varepsilon^{0.6}$ MPa in tension and in compression.

PLAN

We substitute the strain value in Equations (3.36) and obtain the equation for stress in terms of y.

SOLUTION

Substituting the strains in the stress–strain relation in Equations (3.36), we obtain

$$\sigma_{xx} = \begin{cases} 9000\varepsilon_{xx}^{0.6} \text{ MPa} & \varepsilon_{xx} \geq 0 \\ -9000(-\varepsilon_{xx})^{0.6} \text{ MPa} & \varepsilon_{xx} \leq 0 \end{cases} \quad \text{or}$$

$$\sigma_{xx} = \begin{cases} 9000(-0.0125y)^{0.6} \text{ MPa} & y \leq 0 \\ (-9000(0.0125y)^{0.6}) \text{ MPa} & y \geq 0 \end{cases}$$

$$\textbf{ANS.}\quad \sigma_{xx} = \begin{cases} 649.2(-y)^{0.6} \text{ MPa} & y \leq 0 \\ -649.2(y)^{0.6} \text{ MPa} & y \geq 0 \end{cases} \tag{E4}$$

The strains and stresses can be found at different values of y and plotted as shown in Figure 3.49.

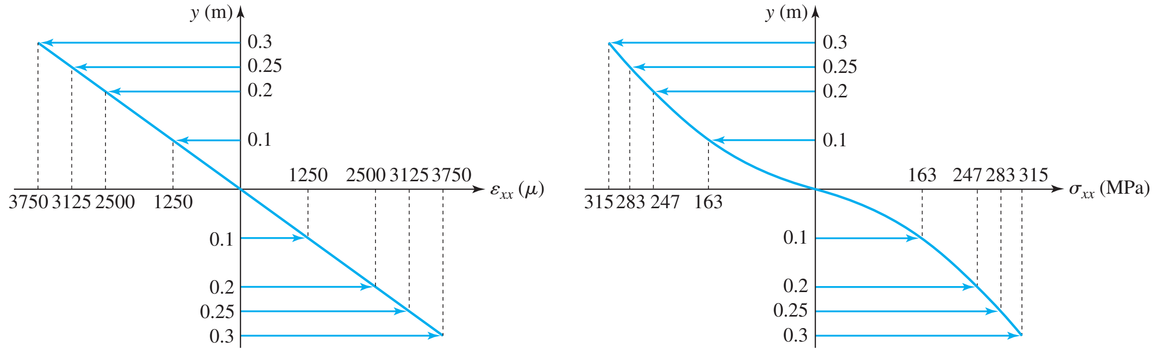


Figure 3.49 Strain and stress distributions in Example 3.17.

COMMENT

1. To better appreciate the stress distribution we can plot it across the entire cross section, as shown in Figure 3.50.

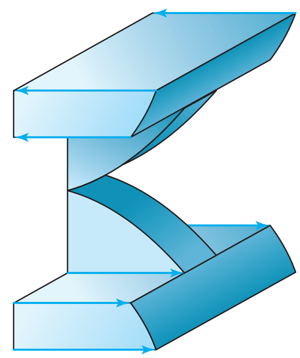


Figure 3.50 Stress distribution across cross section in Example 3.17.

PROBLEM SET 3.4

Fatigue

3.137 A machine component is made from a steel alloy that has an *S–N* curve as shown in Figure 3.36. Estimate the service life of the component if the peak stress is reversed at the rates shown. (a) 40 ksi at 200 cycles per minute. (b) 36 ksi at 250 cycles per minute. (c) 32 ksi at 300 cycles per minute.

3.138 A machine component is made from an aluminum alloy that has an *S–N* curve as shown in Figure 3.36. What should be the maximum permissible peak stress in MPa for the following situations: (a) 17 hours of service at 100 cycles per minute. (b) 40 hours of service at 50 cycles per minute. (c) 80 hours of service at 20 cycles per minute.

3.139 A uniaxial stress acts on an aluminum plate with a hole is shown in Figure P3.139. The aluminum has an S–N curve as shown in Figure 3.36. Predict the number of cycles the plate could be used if d = 3.2 in. and the far-field stress σ = 6 ksi.

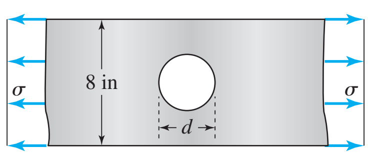


Figure P3.139

3.140 A uniaxial stress acts on an aluminum plate with a hole is shown in Figure P3.139. The aluminum has an S–N curve as shown in Figure 3.36. Determine the maximum diameter of the hole to the nearest $\frac{1}{8}$ in., if the predicted service life of one-half million cycles is desired for a uniform far-field stress $\sigma = 6$ ksi.

3.141 A uniaxial stress acts on an aluminum plate with a hole is shown in Figure P3.139. The aluminum has an S–N curve as shown in Figure 3.36. Determine the maximum far-field stress σ if the diameter of the hole is 2.4 in. and a predicted service life of three-quarters of a million cycles is desired.

Nonlinear material models

3.142 Bronze has a yield stress σ_{yield} = 18 ksi in tension, a yield strain ε_{yield} = 0.0012, ultimate stress σ_{ult} = 50 ksi, and the corresponding ultimate strain ε_{ult} = 0.50. Determine the material constants and plot the resulting stress–strain curve for (a) the elastic–perfectly plastic model. (b) the linear strain-hardening model. (c) the nonlinear power-law model.

3.143 Cast iron has a yield stress σ_{yield} = 220 MPa in tension, a yield strain ε_{yield} = 0.00125, ultimate stress σ_{ult} = 340 MPa, and the corresponding ultimate strain ε_{ult} = 0.20. Determine the material constants and plot the resulting stress–strain curve for: (a) the elastic–perfectly plastic model. (b) the linear strain-hardening model. (c) the nonlinear power-law model.

3.144 A solid circular shaft of 3-in. diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho \times 10^{-3}$, where ρ is the radial coordinate measured in inches. The shaft is made from an elastic–perfectly plastic material, which has a yield stress τ_{yield} = 18 ksi and a shear modulus G = 12,000 ksi. Write the expressions for $\tau_{x\theta}$ as a function of ρ and plot the shear strain $\gamma_{x\theta}$ and shear stress $\tau_{x\theta}$ distributions across the cross section.

3.145 A solid circular shaft of 3-in. diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho \times 10^{-3}$, where ρ is the radial coordinate measured in inches.The shaft is made form a bilinear material as shown in Figure 3.40. The material has a yield stress τ_{yield} = 18 ksi and shear moduli G_1 = 12,000 ksi and G_2 = 4800 ksi. Write the expressions for $\tau_{x\theta}$ as a function of ρ and plot the shear strain $\gamma_{x\theta}$ and shear stress $\tau_{x\theta}$ distributions across the cross section.

3.146 A solid circular shaft of 3-in. diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho \times 10^{-3}$, where ρ is the radial coordinate measured in inches.The shaft material has a stress–strain relationship given by $\tau = 243\gamma^{0.4}$ ksi. Write the expressions for $\tau_{x\theta}$ as a function of ρ and plot the shear strain $\gamma_{x\theta}$ and shear stress $\tau_{x\theta}$ distributions across the cross section.

3.147 A solid circular shaft of 3-in diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho \times 10^{-3}$, where ρ is the radial coordinate measured in inches. The shaft material has a stress–strain relationship given by $\tau = 12{,}000\gamma - 120{,}000\gamma^2$ ksi. Write the expressions for $\tau_{x\theta}$ as a function of ρ and plot the shear strain $\gamma_{x\theta}$ and shear stress τ_{xq} distributions across the cross section.

3.148 A hollow circular shaft has an inner diameter of 50 mm and an outside diameter of 100 mm. The shear strain at a section in polar coordinates was found to be $\gamma_{x\theta} = 0.2\rho$, where ρ is the radial coordinate measured in meters.The shaft is made from an elastic–perfectly plastic material that has a shear yield stress τ_{yield} = 175 MPa and a shear modulus G = 26 GPa. Write the expressions for τ_{xq} as a function of ρ and plot the shear strain γ_{xq} and shear stress τ_{xq} distributions across the cross section.

3.149 A hollow circular shaft has an inner diameter of 50 mm and an outside diameter of 100 mm. The shear strain at a section in polar coordinates was found to be $\gamma_{x\theta} = 0.2\rho$, where ρ is the radial coordinate measured in meters. The shaft is made from a bilinear material as shown in Figure 3.40. The material has a shear yield stress τ_{yield} = 175 MPa and shear moduli G_1 = 26 GPa and G_2 = 14 GPa. Write the expressions for τ_{xq} as a function of ρ and plot the shear strain γ_{xq} and shear stress τ_{xq} distributions across the cross section.

3.150 A hollow circular shaft has an inner diameter of 50 mm and an outside diameter of 100 mm. The shear strain at a section in polar coordinates was found to be $\gamma_{x\theta} = 0.2\rho$, where ρ is the radial coordinate measured in meters. The shaft material has a stress–strain relationship given by $\tau = 3435\gamma^{0.6}$ MPa. Write the expressions for τ_{xq} as a function of ρ and plot the shear strain γ_{xq} and shear stress τ_{xq} distributions across the cross section.

3.151 A hollow circular shaft has an inner diameter of 50 mm and an outside diameter of 100 mm. The shear strain at a section in polar coordinates was found to be $\gamma_{x\theta} = 0.2\rho$, where ρ is the radial coordinate measured in meters. The shaft material has a stress–strain relationship given by $\tau = 26{,}000\gamma - 208{,}000\gamma^2$ MPa. Write the expressions for τ_{xq} as a function of ρ and plot the shear strain γ_{xq} and shear stress τ_{xq} distributions across the cross section.

3.152 A rectangular beam has the dimensions shown in Figure P3.152. The normal strain due to bending about the z axis was found to vary as $\varepsilon_{xx} = -0.01y$, with y measured in meters. The beam is made from an elastic–perfectly plastic material that has a yield stress $\sigma_{yield} = 250$ MPa and a modulus of elasticity $E = 200$ GPa. Write the expressions for normal stress σ_{xx} as a function of y and plot the σ_{xx} distribution across the cross section. Assume similar material behavior in tension and compression

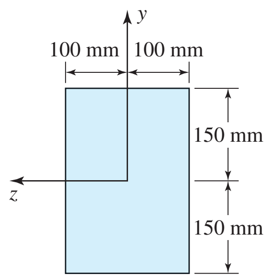


Figure P3.152

3.153 A rectangular beam has the dimensions shown in Figure P3.152. The normal strain due to bending about the z axis was found to vary as $\varepsilon_{xx} = -0.01y$, with y measured in meters. The beam is made from a bilinear material as shown in Figure 3.40. The material has a yield stress $\sigma_{yield} = 250$ MPa and moduli of elasticity $E_1 = 200$ GPa and $E_2 = 80$ GPa. Write the expressions for normal stress σ_{xx} as a function of y and plot the σ_{xx} distribution across the cross section. Assume similar material behavior in tension and compression.

3.154 A rectangular beam has the dimensions shown in Figure P3.152. The normal strain due to bending about the z axis was found to vary as $\varepsilon_{xx} = -0.01y$, with y measured in meters. The beam material has a stress–strain relationship given by $\sigma = 952\varepsilon^{0.2}$ MPa. Write the expressions for normal stress σ_{xx} as a function of y and plot the σ_{xx} distribution across the cross section. Assume similar material behavior in tension and compression.

3.155 A rectangular beam has the dimensions shown in Figure P3.152. The normal strain due to bending about the z axis was found to vary as $\varepsilon_{xx} = -0.01y$, with y measured in meters. The beam material has a stress–strain relationship given by $\sigma = 200\varepsilon - 2000\varepsilon^2$ MPa. Write the expressions for normal stress σ_{xx} as a function of y and plot the σ_{xx} distribution across the cross section. Assume similar material behavior in tension and compression.

3.12* CONCEPT CONNECTOR

Several pioneers concluded that formulas in the mechanics of materials depend on quantities that must be measured experimentally yet Thomas Young is given credit for discovering the modulus of elasticity. History also shows that there was a great controversy over the minimum number of independent constants needed to describe the linear relationship between stress and strain. The controversy took 80 years to resolve. Experimental data were repeatedly explained away when they did not support the prevalent theories at that time. Section 3.12.1 describes the vagaries of history in giving credit and the controversy over material constants.

Composite structural members are a growing area of application of mechanics of materials. Fishing rods, bicycle frames, the wings and control surfaces of aircrafts, tennis racquets, boat hulls, storage tanks, reinforced concrete bars, wooden beams stiffened with steel, laminated shafts, and fiberglass automobile bodies are just some examples. Section 3.12.2 discusses material grouping as a prelude to the discussion of composites in Section 3.12.3

3.12.1 History: Material Constants

Even as a child, Robert Hooke (1635-1703) took an interest in mechanical toys and drawings. In 1662 he was appointed curator of experiments for the Royal Society in England, thanks to his inventive abilities and his willingness to design apparatus in pursuit of ideas from other Society fellow as well as his own. A skilled architect and surveyor, he assisted Christopher Wren on the city that rose up again after the Great Fire of London. He left his mark as well in optics, astronomy, and biology, and indeed he coined the word *cell*. Among his many works are experiments on springs and elastic bodies. In 1678 he published his results, including the linear relationship between force and deformation now known as Hooke's law. In his words, "*Ut tensio sic vis*": as the extension, so is the force. (In 1680 in France, while conducting experiments with beams, Edme Mariotte arrived at the same linear relationship independently.) In acknowledgment of his work on elastic bodies, the stress–strain relation of Equation (3.1) is also called Hooke's law.

Leonard Euler (1707-1783), in his mathematical studies on beam buckling published in 1757, also used Hooke's law and introduced what he called the *moment of stiffness*. He suggested that this moment of stiffness could be determined experimentally. This moment of stiffness is the bending rigidity, which we will study in Chapter 6 on beam bending. It was Giordano Ricardi who proposed the idea that material constant must be determined experimentally. In 1782 he described the first six modes of vibration for chimes of brass and steel, giving values for the modulus of elasticity. Credit for defining and measuring the modulus of elasticity, however, is given to Thomas Young (1773-1829), and is often referred to as Young's modulus.

After Young resigned in 1803 from the Royal Institute in England, where he had been professor of natural philosophy, he published his course material. Here he defines the modulus of elasticity in terms of the pressure produced at the base of a column of given cross-sectional area due to its own weight. This definition includes the area of cross section, which is like the axial rigidity we will study in chapter 4 on axial members. The definition of the modulus of elasticity as purely a material property—independent of geometry—is a later development.

Thomas Young

Simeon Poisson

Figure 3.51 Constants named for these pioneers.

Poisson's ratio is part of a controversy that raged over most of the 19th century. The molecular theory of stress, initiated by the French engineer and physicist Claude-Louis Navier (1785-1836), is based on the central-force concept described in the Section 1.5. Navier himself derived the equilibrium equations, in terms of displacement, using this theory, but with only one independent material constant for isotropic bodies. In 1839 George Green started from the alternative viewpoint that at equilibrium the potential energy must be minimum. Green came to the conclusion that there must be two independent constants for the isotropic stress–strain relationship. From his own independent analysis, also using Navier's molecular theory of stress, Poisson had concluded that the Poisson's ratio must be 1/4. With this value of Poisson's ratio, the equilibrium equations of Green and Navier become identical. While Guillaume Wertheim's experiments on glass and brass in 1848 did not support this value, Wertheim continued to believe that only one independent material constant was needed.

Believers in the existence of one independent material constant dismissed experimental results on the basis that the materials on which the experiment was conducted were not truly isotropic. In the case of anisotropic material, Cauchy and Poisson (using Navier's molecular theory of stress) concluded that there were fifteen independent material constants, whereas Green's analysis showed that twenty-one independent material constants relate stress to strain. The two viewpoints could be resolved if there were six relationships between the material constants. Woldemar Voigt's experiments between 1887 and 1889 on single crystals with known anisotropic properties showed that the six relationships between the material constants were untenable. Nearly half a century after the deaths of Navier, Poisson, Cauchy, and Green, his experimental results finally resolved the controversy. Today we accept that isotropic materials have two independent constants in the general linear stress–strain relationship, whereas anisotropic materials have twenty-one.

3.12.2 Material Groups

There are thirty-one types of crystals. Bodies made up of these crystals can be grouped into classes, based on the independent material constants in the linear stress–strain relationship. The most general anisotropic material, which requires twenty-one independent constants, is also called *triclinic* material. Three other important nonisotropic material groups are monoclinic, orthotropic, and transversely isotropic materials.

- *Monoclinic* materials require thirteen independent material constants. Here the z plane is the plane of symmetry. This implies that the stress–strain relationships are the same in the positive and negative z directions.
- *Orthotropic* materials require nine independent constants. Orthotropic materials have two orthogonal planes of symmetry. In other words, if we rotate the material by 90° about the x or the y axis, we obtain the same stress–strain relationships.
- *Transversely isotropic* materials require five independent material constants. Transversely isotropic materials are isotropic in a plane. In other words, rotation by an arbitrary angle about the z axis does not change the stress–strain relationship, and the material is isotropic in the xy plane.
- *Isotropic* materials require only two independent material constants. Rotation about the x, y, or z axis by any arbitrary angle results in the same stress–strain relationship.

3.12.3 Composite Materials

A body made from more than one material can be called a *composite*. The ancient Egyptians made composite bricks for building the pyramids by mixing straw and mud. The resulting brick was stronger than the brick made from mud alone. Modern polymer composites rely on the same phenomenological effect in mixing fibers and epoxies to get high strength and high stiffness per unit weight.

Fibers are inherently stiffer and stronger than bulk material. Bulk glass such as in window panes has a breaking strength of a few thousand psi. Glass fibers, however, have a breaking strength on the order of one-half million psi. The increase in strength and stiffness is due to a reduction of defects and the alignment of crystals along the fiber axis. The plastic epoxy holds these high-strength and high-stiffness fibers together.

In long or continuous-fiber composites, a *lamina* is constructed by laying the fibers in a given direction and pouring epoxy on top. Clearly each lamina will have different mechanical properties in the direction of the fibers and in the direction perpendicular to the fibers. If the properties of the fibers and the epoxies are averaged (or *homogenized*), then each lamina can be regarded as an orthotropic material. Laminae with different fiber orientations are then put together to create a *laminate*, as shown in Figure 3.52. The overall properties of the laminate can be controlled by the orientation of the fibers and the stacking sequence of the laminae. The designer thus has additional design variables, and material properties can thus be tailored to the design requirements. Continuous-fiber composite technology is still very expensive compared to that of metals, but a significant weight reduction justifies its use in the aerospace industry and in specialty sports equipment.

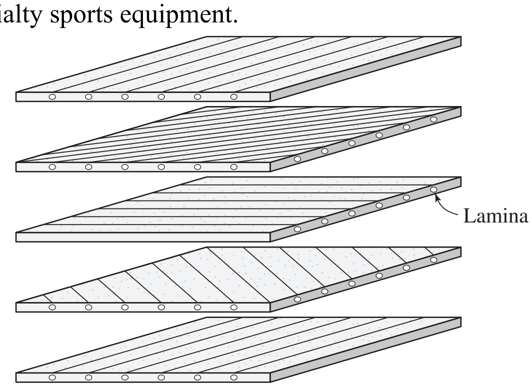


Figure 3.52 Laminate construction.

One way of producing short-fiber composites is to spray fibers onto epoxy and cure the mixture. The random orientation of the fibers results in an overall transversely isotropic material whose properties depend on the ratio of the volume of fibers to the volume of epoxy. Chopped fibers are cheaper to produce than the continuous-fiber composites and are finding increasing use in automobile and marine industries for designing secondary structures, such as body panels.

3.13 CHAPTER CONNECTOR

In this chapter we studied the many ways to describe material behavior and established the empirical relationships between stress and strain. We saw that the number of material constants needed depends on the material model we wish to incorporate into our analysis. The simplest material model is the linear, elastic, isotropic material that requires only two material constants.

We also studied how material models can be integrated into a logic by which we can relate displacements to external forces. A more complex material model changes the stress distribution across the cross section, but not the key equations—the relationships between displacements and strains or between stresses and internal forces and moments. Similarly, we can add complexity to the relationship between displacements and strains without changing the material model. Thus the modular structure of the logic permits us to add complexities at several points, then carry the complexity forward into the equations that are otherwise unchanged.

Chapters 4 through 7 will extend the logic shown in Figure 3.15 to axial members, torsion of circular shafts, and symmetric bending of beams. The idea is to develop the simplest possible theories for these structural members. To do so, we shall impose limitations and make assumptions regarding loading, material behavior, and geometry. The difference between limitations and assumptions is in the degree to which the theory must be modified when a limitation or assumption is not valid. An entire theory must be redeveloped if a limitation is to be overcome. In contrast, assumptions are points where complexities can be added, and the derivation path that was established for the simplified theory can then be repeated. Examples, problems, and optional sections will demonstrate the addition of complexities to the simplified theories.

All these theories will have certain limitations in common:

1. The length of the member is significantly (approximately 10 times) greater than the greatest dimension in the cross section. Approximations across the cross section are now possible, as the region of approximation is small. We will deduce constant or linear approximations of deformation across the cross section and confirm the validity of an approximation from photographs of deformed shapes.
2. We are away from regions of stress concentration, where displacements and stresses can be three-dimensional. The results from the simplified theories can be extrapolated into the region of stress concentration, as described in Section 3.7.
3. The variation of external loads or change in cross-sectional area is gradual, except in regions of stress concentration. The theory of elasticity shows that this limitation is necessary; otherwise the approximations across the cross section would be untenable.
4. The external loads are such that axial, torsion, and bending can be studied individually. This requires not only that the applied loads be in a given direction, but also that the loads pass through a specific point on the cross section.

Often reality is more complex than even the most sophisticated theory can explain, and the relationship between variables must be modified empirically. These empirically modified formulas of mechanics of materials form the basis of most structural and machine design.

POINTS AND FORMULAS TO REMEMBER

- The point up to which stress and strain are linearly related is called proportional limit.
- The largest stress in the stress–strain curve is called ultimate stress.
- The sudden decrease in the cross-sectional area after ultimate stress is called necking.
- The stress at the point of rupture is called fracture or rupture stress.
- The region of the stress–strain curve in which the material returns to the undeformed state when applied forces are removed is called elastic region.
- The region in which the material deforms permanently is called plastic region.
- The point demarcating the elastic from the plastic region is called yield point.
- The stress at yield point is called yield stress.
- The permanent strain when stresses are zero is called plastic strain.
- The offset yield stress is a stress that would produce a plastic strain corresponding to the specified offset strain.
- A material that can undergo large plastic deformation before fracture is called ductile material.
- A material that exhibits little or no plastic deformation at failure is called brittle material.
- Hardness is the resistance to indentation.
- Raising the yield point with increasing strain is called strain hardening.
- A ductile material usually yields when the maximum shear stress exceeds the yield shear stress of the material.
- A brittle material usually ruptures when the maximum tensile normal stress exceeds the ultimate tensile stress of the material.
- $\sigma = E\varepsilon$ (3.1) $\qquad \nu = -\left(\dfrac{\varepsilon_{\text{lateral}}}{\varepsilon_{\text{longitudinal}}}\right)$ (3.2) $\qquad \tau = G\gamma$ (3.3)
- where E is the modulus of elasticity, ν is Poisson's ratio, and G is the shear modulus of elasticity.
- The slope of the tangent to the stress–strain curve at a given stress value is called tangent modulus.
- The slope of the line that joins the origin to the given stress value is called secant modulus.
- Failure implies that a component or a structure does not perform the function for which it was designed.
- Failure could be due to too little or too much deformation or strength.
- Factor of safety: $K_{\text{safety}} = \dfrac{\text{failure-producing value}}{\text{computed (allowable) value}}$ (3.10)
- The factor of safety must always be greater than 1.
- The failure-producing value could be the value of deformation, yield stress, ultimate stress, or loads on a structure.
- An isotropic material has a stress–strain relation that is independent of the orientation of the coordinate system.
- In a homogeneous material the material constants do not change with the coordinates x, y, or z of a point.
- There are only two independent material constants in a linear stress–strain relationship for an isotropic material, but there can be 21 independent material constants in an anisotropic material.
- Generalized Hooke's law for isotropic materials:
- $$\begin{aligned} \varepsilon_{xx} &= [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]/E & \gamma_{xy} &= \tau_{xy}/G \\ \varepsilon_{yy} &= [\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})]/E & \gamma_{yz} &= \tau_{yz}/G \\ \varepsilon_{zz} &= [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]/E & \gamma_{zx} &= \tau_{zx}/G \end{aligned} \qquad G = \frac{E}{2(1+\nu)}$$ (3.14a) through (3.14f)